Another Look at Gaussian CGS Units

or, Why CGS Units Make You Cool

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Abstract

In this paper, I compare the merits of Gaussian CGS and SI units in a variety of different scenarios for problems involving electricity and magnetism. I conclude that in most (though not all) situations, using Gaussian CGS units makes the deeper meanings of most formulas much more lucid than using SI.

1 Electrostatics

In SI units (hereafter referred to simply as "SI"), Coulomb's law reads

$$\mathbf{F}_{\mathrm{SI}} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{1}$$

while in Gaussian CGS units (hereafter referred to simply as "CGS"), it reads

$$\mathbf{F}_{\mathrm{CGS}} = \frac{q_1 q_2}{r^2}\hat{\mathbf{r}}. \tag{2}$$

The Gaussian definition is eminently sensible and easily reproducible. It states that two charges of magnitude 1 esu each separated by 1 cm will exert a force of magnitude 1 dyn on each other. It can be experimentally duplicated and verified, and from this other quantities like the charge of an electron can be found.

The SI definition is an artifact of 19^{th}-century bickering over what to call the prefactor in front of Coulomb's law. It was eventually decided that to rationalize Gauss's law which is more fundamental than Coulomb's law, the prefactor should be $\frac{1}{4\pi\epsilon_0}$. It is often said that the permittivity of free space can be justified by quantum field theory in that the coming and going of virtual particles in a vacuum (or something like that – I don't really know anything about quantum field theory beyond that) is what really allows the vacuum to support an electric field. But frankly, the choice is arbitrary, because it can just as easily be argued that the prefactor in front of Coulomb's law being equal to unity rather than zero suggests a nice maximum value for the force between two point charges.

Neither system has an advantage when it comes to Gauss's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \text{ (SI)} \tag{3}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \text{ (CGS)}. \tag{4}$$

In either case, there are prefactors before the charge density on the right-hand side of the equation that do not really have much meaning other than serving as simple proportionality constants. This is where the Lorentz-Heaviside CGS unit system really shines, because its form of Gauss's law is

$$\nabla \cdot \mathbf{E} = \rho \tag{5}$$

and its form of Coulomb's law is (in terms of the electric field of a point charge source)

$$\mathbf{E} = \frac{q}{4\pi r^2}\hat{\mathbf{r}}. \tag{6}$$

This shows clearly that the electric flux through an arbitrary surface is exactly equal to the charge enclosed in that surface, and the electric field of a point charge on a fictitious sphere of radius r is simply the average surface charge density as seen on the surface of that sphere. There are no other prefactors cluttering these equations that need to be justified.

Section score: Gaussian CGS – 1, Lorentz-Heaviside CGS – 1, SI – 0.

(For the record, this is one of the few sections where Lorentz-Heaviside units will be discussed. From now on, "CGS" units will mostly mean Gaussian CGS units.)

2 Capacitance

It is important to recognize that while capacitance in SI has dimensions relating to current, length, mass, and time all together, capacitance in CGS has dimensions equal to length and nothing more; sometimes, the CGS dimensions of capacitance as length presents an advantage, while other times it presents a disadvantage.

For example, let us consider a concentric spherical capacitor with inner radius r_1 and outer radius r_2. The capacitance of this system is

$$C = \frac{4\pi\epsilon_0}{\frac{1}{r_1} - \frac{1}{r_2}} \text{ (SI)} \tag{7}$$

$$C = \frac{1}{\frac{1}{r_1} - \frac{1}{r_2}} \text{ (CGS).} \tag{8}$$

The advantage CGS has here is that there are no extra prefactors needlessly cluttering an already messy equation. The prefactors in SI present no new meaning and simply detract from the ability to do the calculation quickly.

The advantage of using CGS with spherical capacitors becomes much more apparent when considering the capacitance of a single charged spherical shell of radius r relative to an outer shell that can be considered to be infinitely large:

$$C = 4\pi\epsilon_0 r \text{ (SI)} \tag{9}$$

$$C = r \text{ (CGS).} \tag{10}$$

Here, the capacitance of such a sphere is exactly equal to its radius in CGS; in SI, the prefactors once again present no new information.

The advantage of Gaussian CGS quickly goes away, though, when considering a parallel-plate capacitor of area $\mathcal{S}$ and separation s:

$$C = \frac{\epsilon_0 \mathcal{S}}{s} \text{ (SI)} \tag{11}$$

$$C = \frac{\mathcal{S}}{4\pi s} \text{ (CGS).} \tag{12}$$

Because most pedagogical discussions of electric permittivity begin with capacitors filled with dielectric materials, I feel like the presence of the permittivity of free space in SI is more meaningful here than in Coulomb's law. Plus, in CGS, the prefactor $\frac{1}{4\pi}$ does not add any new meaning. The real winner, though, would be Lorentz-Heaviside CGS units

$$C = \frac{\mathcal{S}}{s} \tag{13}$$

because the capacitance would just be the ratio of the area to the length without any other prefactors. Lorentz-Heaviside units would not work though for other situations, such as the aforementioned spherical capacitor.

For other capacitors, like a cylindrical capacitor with inner radius r_1 and outer radius r_2, there is no real advantage to using one system or the other

$$\frac{dC}{d\ell} = \frac{2\pi\epsilon_0}{\ln\left(\frac{r_2}{r_1}\right)} \text{ (SI)} \tag{14}$$

$$\frac{dC}{d\ell} = \frac{1}{2\ln\left(\frac{r_2}{r_1}\right)} \text{ (CGS)} \tag{15}$$

because the geometry means that extra factors of $\frac{1}{2}$ or 2 pop in one way or the other.

Section score: Gaussian CGS – 1, Lorentz-Heaviside CGS – 1, SI – 0.

3 Relativity

Relativity is really the area where CGS shines compared to SI. In relativity, when considering electricity and magnetism, the key idea is that those fields are equivalent to one another in different frames of reference. In CGS, it is in fact possible to freely add and subtract electric and magnetic fields, whereas in SI, prefactors of the speed of light c must be used too.

The electromagnetic field tensor is

$$F^{\mu\nu} = \begin{bmatrix} 0 & \frac{(\mathbf{E})^x}{c} & \frac{(\mathbf{E})^y}{c} & \frac{(\mathbf{E})^z}{c} \\ -\frac{(\mathbf{E})^x}{c} & 0 & (\mathbf{B})^z & -(\mathbf{B})^y \\ -\frac{(\mathbf{E})^y}{c} & -(\mathbf{B})^z & 0 & (\mathbf{B})^x \\ -\frac{(\mathbf{E})^z}{c} & (\mathbf{B})^y & -(\mathbf{B})^x & 0 \end{bmatrix} \text{ (SI)} \tag{16}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & (\mathbf{E})^x & (\mathbf{E})^y & (\mathbf{E})^z \\ -(\mathbf{E})^x & 0 & (\mathbf{B})^z & -(\mathbf{B})^y \\ -(\mathbf{E})^y & -(\mathbf{B})^z & 0 & (\mathbf{B})^x \\ -(\mathbf{E})^z & (\mathbf{B})^y & -(\mathbf{B})^x & 0 \end{bmatrix} \text{ (CGS).} \tag{17}$$

In special relativity, it is usually desirable to have all the components of a geometric object have the same dimensions; with CGS, that is much easier than with SI because **E** and **B** already have the same dimensions in CGS.

This also leads to the invariant quantities associated with those fields,

$$\sum_{\mu,\nu=0}^{3} F^{\mu\nu} F_{\mu\nu} = 2\left(\mathbf{B}^2 - \frac{\mathbf{E}^2}{c^2}\right) \text{ (SI)} \tag{18}$$

$$\sum_{\mu,\nu=0}^{3} F^{\mu\nu} F_{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right) \text{ (CGS)} \tag{19}$$

and as is immediately apparent, CGS allows for the free addition and subtraction of electric and magnetic fields where SI does not, solidifying the notion that they are equivalent fields in all senses of the word as viewed in different frames of reference.

In special relativity, because the speed of light is the speed limit, the velocity $\mathbf{v}$ is eschewed in favor of the quantity $\beta = \frac{\mathbf{v}}{c}$. Given that, the transformation laws for the electric and magnetic fields and the Lorentz force law make it abundantly clear which unit system is more relativity-friendly out-of-the-box. In SI, the transformation and Lorentz force laws are

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \tag{20}$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \tag{21}$$

$$\mathbf{E}'_{\perp} = \gamma \cdot (\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}) \tag{22}$$

$$\mathbf{B}'_{\perp} = \gamma \cdot (\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_{\perp}) \tag{23}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{24}$$

while in CGS they are

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \tag{25}$$
$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \tag{26}$$
$$\mathbf{E}'_{\perp} = \gamma \cdot (\mathbf{E}_{\perp} + \beta \times \mathbf{B}_{\perp}) \tag{27}$$
$$\mathbf{B}'_{\perp} = \gamma \cdot (\mathbf{B}_{\perp} - \beta \times \mathbf{E}_{\perp}) \tag{28}$$
$$\mathbf{F} = q(\mathbf{E} + \beta \times \mathbf{B}). \tag{29}$$

One other thing to note is that the relative weakness of the magnetic contribution to the Lorentz force is immediately apparent in CGS because usually β is much smaller than 1; this cannot really be gleaned from SI.

Finally, it is important to note that the electromagnetic four-potential is defined differently in each system:

$$A^{\mu} = (\frac{\phi}{c}, \mathbf{A}) \text{ (SI)} \tag{30}$$
$$A^{\mu} = (\phi, \mathbf{A}) \text{ (CGS).} \tag{31}$$

CGS thus hews closer to the convention of special relativity to make all the components of a geometric object have the same dimensions, because ϕ and A already have the same dimensions, whereas extra factors of c are needed in SI to achieve this.

The greater number of symmetries is apparent, and it is clear that CGS rules the day as far as electrodynamics in special relativity is concerned.

Section score: CGS – 4, SI – 0.

4 Magnetism

The Biot-Savart law, describing the magnetic field created by a moving point charge, is

$$\mathbf{B} = \frac{\mu_0 q \mathbf{v} \times \hat{\mathbf{r}}}{4\pi r^2} \text{ (SI)} \tag{32}$$
$$\mathbf{B} = \frac{q\beta \times \hat{\mathbf{r}}}{r^2} \text{ (CGS).} \tag{33}$$

In either case, it can be seen that the magnetic field is quite weak. But it is far easier to see why in CGS. In SI, the factor $\frac{\mu_0}{4\pi}$ is quite small, but it is just a prefactor so the meaning gets obfuscated. In CGS, though, it is obvious that the reason why magnetic fields are usually quite small is because a charged particle would have to be moving very fast for a significant magnetic field to be produced (at which point the Biot-Savart law would require relativistic corrections), and most problems consider slower particles anyway. Plus, it is easy to compare the weakness of the magnetic field to the strength of the electric field; such a comparison would be absurd in SI because those fields have different dimensions, so a comparison would be akin to that of apples and oranges.

In Ampère's law, no system has the advantage, because all of them convey essentially the same information, and all of them have a few superfluous prefactors present:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{c^2\, \partial t} \text{ (SI)} \tag{34}$$

$$\nabla \times \mathbf{B} = \frac{4\pi \mathbf{J}}{c} + \frac{\partial \mathbf{E}}{c\, \partial t} \text{ (CGS).} \tag{35}$$

However, when examining Faraday's law, CGS has the advantage because for all the terms not containing ∇, there is a prefactor of $\frac{1}{c}$ which makes for some nice symmetry after having examined Ampère's law that is not present in SI:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ (SI)} \tag{36}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{c\, \partial t} \text{ (CGS).} \tag{37}$$

Section score: CGS – 2, SI – 0.

5 Waves

Waves are another area where CGS is helpful. When describing an electromagnetic wave, in CGS statements like

$$|\mathbf{E}| = |\mathbf{B}| \tag{38}$$

can be made, showing the equivalence of the electric and magnetic fields, whereas other factors of c would be required in SI.

Furthermore, the impedance of free space is

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\ \Omega \text{ (SI)} \tag{39}$$

$$Z_0 = \frac{4\pi}{c} \text{ (CGS).} \tag{40}$$

It is much easier to check for dimensional correctness when working with the impedance of free space in CGS, because impedance has the dimensions of the reciprocal of velocity. Furthermore, it is much easier to remember something like $\frac{4\pi}{c}$ than a precise number like 377 Ω.

This also plays into the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \text{ (SI)} \tag{41}$$

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{1}{Z_0} \mathbf{E} \times \mathbf{B} \text{ (CGS).} \tag{42}$$

In SI, it can be said that the presence of μ_0 in $\mathbf{S}$ indicates that the vacuum can support magnetic fields. But then, why is there not any explicit ϵ_0 to indicate the vacuum's ability to support electric fields? By contrast, the impedance of free space speaks of the vacuum's ability to support both electric and magnetic fields traveling as waves, and this is made all the more clear with CGS.

Finally, establishing Z_0 in CGS allows for writing Ampère's law in the following form:

$$\oint \mathbf{B} \cdot d\ell = Z_0 I. \tag{43}$$

While the presence of Z_0 does not present any new understanding of this law, it does present a nicer symmetry when compared with the voltage across a generalized impedance:

$$V = -\int \mathbf{E} \cdot d\mathbf{s} = ZI. \tag{44}$$

The first equation says the circulation of a magnetic field around a wire is equal to the product of the current and the impedance of the surrounding region. The second equation says that the potential difference between ends of an impedance in a wire is equal to the product of the current through that wire and the impedance itself. That kind of symmetry cannot be achieved in SI.

Section score: CGS – 3, SI – 0.

6 Material media

In CGS, the electric and magnetic fields, electric displacement and magnetizing fields, and the polarization density and magnetization all have the same dimensions, so they can be compared freely. This is not possible in SI without other factors.

Furthermore, comparing the microscopic and macroscopic Maxwell equations in CGS yields only replacing ρ with ρ_f and $\mathbf{J}$ with $\mathbf{J}_f$ and replacing the fields appropriately; in SI, other constants appear and disappear when going between microscopic and macroscopic versions of Maxwell's equations in material media. When comparing how the equations change in CGS

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{45}$$

$$\text{becomes } \nabla \cdot \mathbf{D} = 4\pi\rho_f \tag{46}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{\partial \mathbf{E}}{c\,\partial t} \tag{47}$$

$$\text{becomes } \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J_f} + \frac{\partial \mathbf{D}}{c\,\partial t} \tag{48}$$

versus SI

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{49}$$

$$\text{becomes } \nabla \cdot \mathbf{D} = \rho_f \tag{50}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{c^2\,\partial t} \tag{51}$$

$$\text{becomes } \nabla \times \mathbf{H} = \mathbf{J_f} + \frac{\partial \mathbf{D}}{\partial t} \tag{52}$$

the greater self-consistency of CGS is obvious.

Also, when considering the polarization density

$$\mathbf{D} = \epsilon_0 \mathbf{E} + P \quad (53)$$

$$\text{where } P = \epsilon_0 \chi_e \mathbf{E} \text{ (SI)} \quad (54)$$

$$\mathbf{D} = \mathbf{E} + 4\pi P \quad (55)$$

$$\text{where } P = \chi_e \mathbf{E} \text{ (CGS)} \quad (56)$$

versus the magnetization

$$\mathbf{B} = \mu_0(\mathbf{H} + M) \quad (57)$$

$$\text{where } M = \chi_m \mathbf{H} \text{ (SI)} \quad (58)$$

$$\mathbf{B} = \mathbf{H} + 4\pi M \quad (59)$$

$$\text{where } M = \chi_m \mathbf{H} \text{ (CGS)} \quad (60)$$

there is symmetry in the definitions of the polarization density and the magnetization in CGS, and that symmetry is preserved in the definitions of the electric displacement and magnetic fields. In SI, though, for some reason the factor of ϵ_0 is embedded in the definition of the polarization density while μ_0 is not embedded in the definition of the magnetization, so there is an asymmetry there that is also preserved in the definitions of the electric displacement and magnetic fields. It seems rather arbitrary to include the permittivity of free space in one definition but not the permeability of free space in another, and CGS does not need to worry about that at all.

Section score: CGS – 3, SI – 0.

7 Words of Caution

It is important to know that it is not possible to blindly force-fit μ_0 and ϵ_0 in CGS based on symmetries with laws in SI. For example, it is tempting based on Gauss's law to define $\epsilon_0 \equiv \frac{1}{4\pi}$, and it is tempting based on Ampère's law to define $\mu_0 \equiv \frac{4\pi}{c}$. However, this will yield the equality $\mu_0 \epsilon_0 c = 1$, which is different from the equality $\mu_0 \epsilon_0 c^2 = 1$ as is required by SI. This will also lead to an incorrect conclusion about the value of Z_0[1] as well.

The aforementioned paper asserts that $Z_0 = 1$ in CGS, which is patently false, because it also claims that $\mu_0 = \epsilon_0 = 1$, so $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 1$. In fact, if $\mu_0 = \frac{4\pi}{c}$, as is true in Gaussian CGS, then $Z_0 = \mu_0$ itself. Furthermore, this implies that Z_0 and c can replace factors of 4π in the CGS versions of Maxwell's equations just as well as they replace μ_0 and ϵ_0 in the SI versions of Maxwell's equations. For example, here is what Maxwell's equations in CGS would look like using Z_0

[1] M. Kitano, "The vacuum impedance and unit systems," arXiv:physics/0607056v2 (2008)

and c:

$$\nabla \cdot \mathbf{E} = cZ_0\rho \tag{61}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{62}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{c\,\partial t} \tag{63}$$

$$\nabla \times \mathbf{B} = Z_0\mathbf{J} + \frac{\partial \mathbf{E}}{c\,\partial t}. \tag{64}$$

In SI, this would be

$$\nabla \cdot \mathbf{E} = cZ_0\rho \tag{65}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{66}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{67}$$

$$\nabla \times \mathbf{B} = \frac{Z_0}{c}\mathbf{J} + \frac{\partial \mathbf{E}}{c^2\,\partial t}. \tag{68}$$

It is immediately apparent that there is now a greater degree of similarity between the two unit systems when replacing 4π, μ_0, and ϵ_0 with Z_0 and c. Thus, the myth that the CGS cannot produce a nice version of Maxwell's equations with the quantities c and Z_0 like SI can, perpetuated by the paper by Kitano, can be put to rest.

Section score: CGS – 1, SI – 1.

8 The Narrative

The traditional narrative regarding the unification of electricity and magnetism along with the study of light is that ϵ_0 was meant to just deal with electrical phenomena, μ_0 was meant to just deal with magnetic phenomena, and c was only supposed to be the speed of light. Therefore, the equation $\mu_0\epsilon_0 c^2 = 1$ was supposed to be a great triumph of science.

The problem is that is only half the story, because SI (or MKS, as it was known before) was not the only unit system being used. Simultaneously, Gaussian CGS (because Lorentz-Heaviside CGS had not been developed yet) was in common currency. At that time, the factor c was just a proportionality constant, and while it had the same dimensions as velocity, it was not until the late 19$^{\text{th}}$ century that the connection was made in CGS as well. Relating the c present as a prefactor in Ampère's and Faraday's laws to the c known as the speed of light in CGS was just as great an achievement as the relation of μ_0 and ϵ_0 to c in SI; sadly, because of SI's prevalence over CGS, this other story is too often overlooked.

Furthermore, now that the c present in Maxwell's equations in CGS is known to be the same as the speed of light, it can be more easily justified that CGS can be adapted more easily to show this unification between electricity and magnetism. In SI, the residual presence of μ_0 and ϵ_0 as separate prefactors still shows that SI implicitly considers electricity and magnetism to be separate. Thus, CGS, which is an older system than SI, has aged far better through the unification of electricity and magnetism and optics.

This aging also relates to the higher compatibility of CGS with relativistic electrodynamics. The paper by Kitano posits that the point is moot because CGS was developed long before the development of the Lorentz transformation and the rest of special relativity. Yet, the fact remains that CGS is undoubtedly more relativity-friendly than SI out-of-the-box. Therefore, this statement by Kitano, rather than denigrating CGS as intended, only serves as a testament to how well CGS can handle new developments in electrodynamics like special relativity and how poorly the newer SI handles it.

Finally, the reason for factors like μ_0 and ϵ_0 in SI stem mostly from convenience: the people setting up the SI standards found it easier to measure the force due to currents in wires upon other wires rather than the force of one charge upon another. The definition of the ampere comes from

$$\frac{dF}{d\ell} = \frac{\mu_0 I_1 I_2}{2\pi r} \tag{69}$$

where

$$\mu_0 \equiv 4\pi \times 10^{-7}\ \frac{\mathrm{N}}{\mathrm{A}^2} \tag{70}$$

so that the ampere can be a fundamental unit independent of other mechanical units (kg, m, s). Thus, the ampere is defined as the amount of current such that two wires each with 1 ampere of current separated by 1 meter will exert a force of 2×10^{-7} newtons per meter. This is actually a fairly reasonable definition, and it does make sense to make a quantity of electrodynamics independent of other mechanical quantities. However, the fact remains that ultimately, electric and magnetic fields arise from charged particles; all charged particles are sources of electric fields, but only moving charged particles (in a given frame of reference) are sources of magnetic fields, so there is no single particle that can produce only a magnetic field without an electric field. Given that, it makes more sense to make the unit of charge in electric phenomena the fundamental unit rather than the unit of current as per magnetic phenomena, and this is exactly what CGS does.

Section score: CGS – 2, SI – 0.

9 Conclusion

Thus, after all this, Gaussian CGS units present a more self-consistent, relativity-friendly, dielectric-friendly, symmetrical set of equations than SI units do, while matching any and all benefits that SI may have.

Total score: CGS – 17, SI – 1.