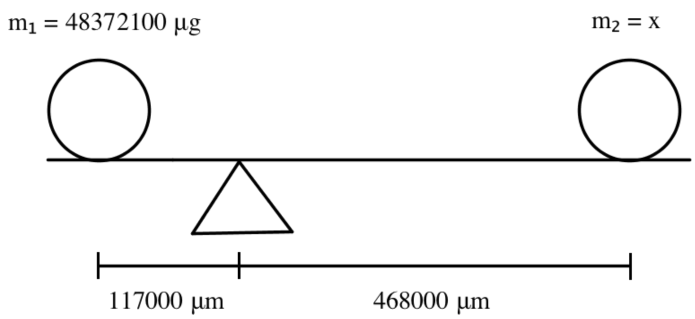
Practice in Theory

This lengthy problem set found in the Atlantean archives suggests that Atlantis was once home to a highly energetic group of theoretical physicists.

1. Phillip is a fisherman who is trying to set up a balance as shown in the picture below: Solve for x, in micrograms, which will balance the scales.



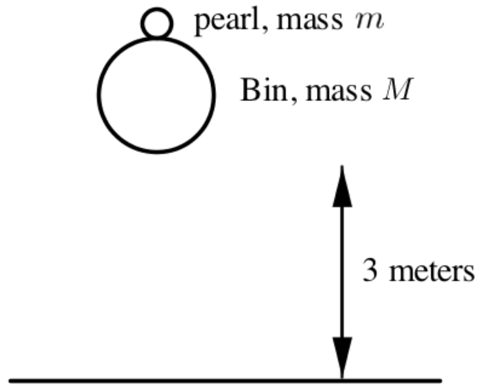
2. Sophia is fishing. She caught a 2.649235000kg herring, but her bucket is 5.623188175m away. Suppose she throws the fish horizontally from a height of 1.524000000m. How fast must she throw the fish in order for it to exactly hit the bottom of the bucket? Express your answer in micrometers per second, rounded to the nearest micrometer per second. Assume $g = 9.8\text{m/s}^2$ exactly.

3. A bunch of fish in a huge barrel with height exactly 40m are trying to avoid being shot by some fishermen frustrated with their recent nautical failures. Led by Johann, their fish leader, the fish decide to gnaw a hole in the side of the barrel to eject themselves out of the hole into a lake located 36.705676571m away. What is the minimum height from the bottom of the barrel at which Johann and his team can gnaw the hole and still land at the water's edge? Express your answer in micrometers, rounded to the nearest micrometer. Assume the fish are ejected at the fluid velocity at the hole.

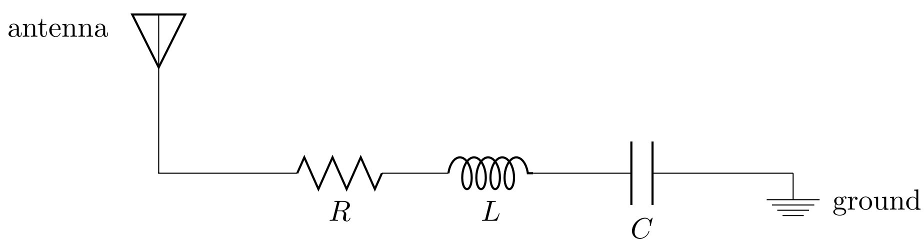
4. Tim is a fat fish who got stuck in the hole in the barrel from the previous problem, and finds himself rolling down a hill of height 27.6566577643m in the barrel. Assume the barrel is a solid cylinder (tightly packed with fish) with radius exactly 2m. What is the angular velocity of Tim's rotation around the center of the barrel once he reaches the bottom of the hill? Express your answer in inverse megaseconds rounded to the nearest inverse megasecond. Assume Tim's mass does not appreciably affect the barrel's motion and that friction due to the motion of the water is negligible. Take the gravitational field strength to be $g = 9.8\text{m/s}^2$ exactly.

5. Bin is a perfectly spherical pufferfish of mass exactly $M = 1$kg who has been cruelly extracted from his tank. His owner places a perfectly spherical pearl of mass $m = 0.20761025117$kg perfectly on top of him so that the centers of the two spheres lie on a vertical line, as shown

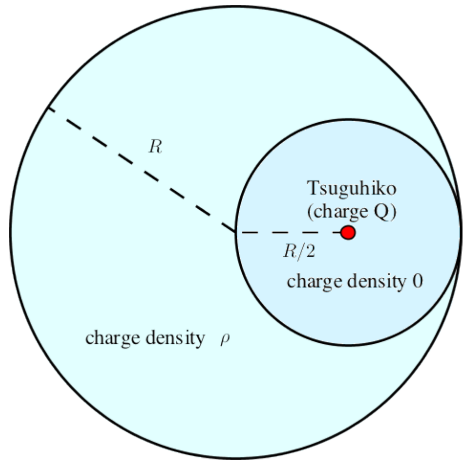
in the figure. He then drops Bin from rest at a height of exactly 3m. Assume all collisions that occur subsequently are elastic. How high does the pearl bounce above its initial position on its first bounce? Express your answer in microns, rounded to the nearest micron.



6. Roberto the whale is trying to listen to his favorite whale-song radio station, but since he is a whale, he cannot buy a radio and so must build an antenna himself. He decides to use a series RLC circuit as shown below. His wires have a total intrinsic resistance of exactly $R = 0.1\Omega$. Since his fins are not so good for coiling wires, he decides to use the smallest L he can. If Roberto wishes to tune to an angular frequency of exactly $\omega = 100$MHz with a quality factor of at least 7.65069672982 (hey, what can you reasonably expect under the sea?), what is the capacitance of the resulting circuit, in femtofarads, rounded to the nearest femtofarad?

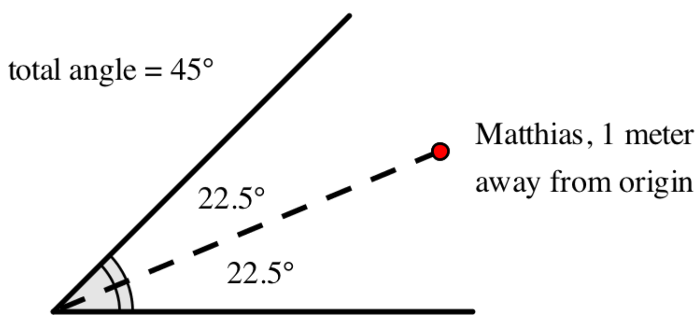


7. Tsuguhiko is a very small electric eel carrying a charge of exactly 1 microcoulomb. He is swimming in a closed, spherical fish tank of radius exactly $R/2 = 0.5$m (filled with uncharged water) which is itself immersed in a larger fish tank of radius exactly $R = 1$m filled with a charged fluid of charge density $\rho = 748.14251529375000\mu\text{C/m}^3$. The small tank Tsuguhiko lives in is positioned so that the surface touches both the center of the larger fish tank and the outside surface of the larger fish tank, as shown in the diagram below. What electric force, in micronewtons, does Tsuguhiko feel as he passes through the center of the fish tank, rounded to the nearest micronewton? The Coulomb's law constant is $1/(4\pi\epsilon_0) = 8.9875517873681764 \times 10^9 \text{Nm}^2/\text{C}^2$.

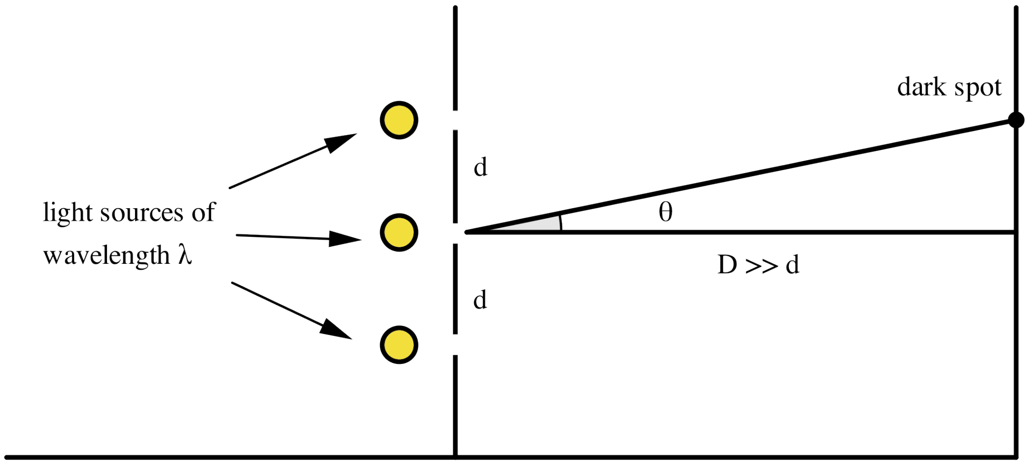


8. Matthias has a massive eel which behaves like a spring of spring constant $K = 452.9269061\text{N/m}$, unstretched length exactly $L = 1\text{m}$, and linear mass density exactly $\lambda = 1\text{kg/m}$. If he suspends it vertically, how far is the midpoint of the spring displaced relative to its unstretched position? Express your answer in nanometers, rounded to the nearest nanometer. Take $g = 9.8\text{m/s}^2$ exactly.

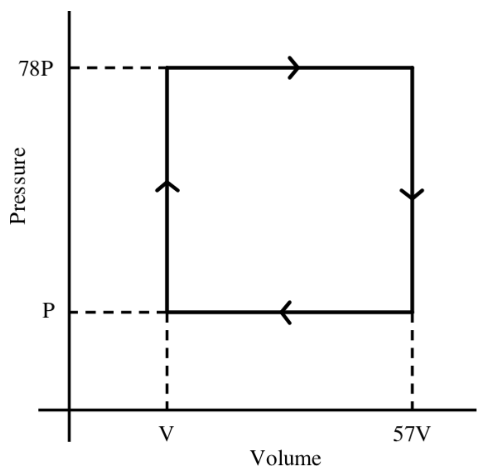
9. Matthias (a different one) is a tiny point-plankton with charge 34.280862935000000 microcoulombs. He has found himself exactly in between two semi-infinite conducting planes which meet at an angle of 45 degrees exactly 1m away from the origin, as shown in the figure below. What is the force on Matthias in micronewtons, rounded to the nearest micronewton? Recall that the Coulomb's law constant is $1/(4\pi\epsilon_0) = 8.9875517873681764 \times 10^9\text{Nm}^2/\text{C}^2$.



10. Herbert and two of his anglerfish friends share a cell in fish prison for deeds too lurid to recount here. They are in a maximum security unit, which has only three tiny slits, regularly spaced with distance $d = 338.8286682\text{nm}$, where they can see out of. In boredom, they each decide to stand in front of a slit and turn on their anglerfish light, which has wavelength exactly $\lambda = 580\text{nm}$ and identical amplitudes. The light travels a distance D very large compared to d until it hits a wall, as shown in the figure. What is the smallest angle θ from the center of the three slits where a dark spot will be observed? Express your answer in microradians, rounded to the nearest microradian.



11. Sergei is building an engine for his submarine. His engine has perfectly reflecting walls and is powered by a gas of photons. Recall that the internal energy U of such a gas is given by $U = \frac{4\sigma}{c}VT^4$, where σ is the Stefan-Boltzmann constant, c is the speed of light, V is the volume, and T is the temperature. His engine subjects the photon gas to the following cycle, where each pair of points is connected by a straight line:



P and V are reference pressures and volumes which Sergei will not tell us. Suppose he supplies his engine with $Q = 41200255.125\text{J}$ worth of heat input and that the engine completely uses all of it. How much work will his engine do?

12. Mario has two distinguishable, one-dimensional quantum fish which are coupled. The system is described by the Hamiltonian:

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(x_1^2 + x_2^2) + gx_1x_2$$

where

$$m = 9.1093821500 \times 10^{-31}\text{kg}$$
$$\omega = 1.0000000000 \times 10^{26}\text{Hz}$$
$$g = 8.6163491022 \times 10^{21}\text{Jm}^{-2}$$

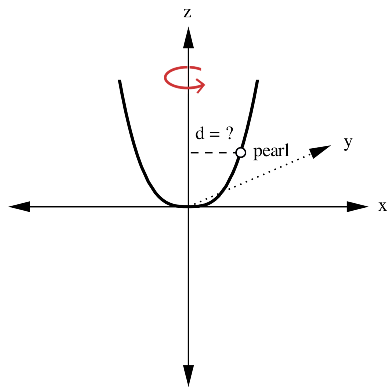
What is the energy of the first excited state of this two-fish system? Express your answer in femtojoules, rounded to the nearest femtojoule. Take $\hbar = 1.0545717260 \times 10^{-34}$Js.

13. Andreas is studying indistinguishable fermion fish ("fermifish") living in a one-dimensional circular fish tank. These fermifish have been bred to have spin 1/2. He has two such fish in a circular tank of circumference exactly $L = 1$m. The fermifish are described by the Hamiltonian:
$$H = \frac{1}{2m}(p_1^2 + p_2^2) - gS_1 \cdot S_2,$$
where $g = 1.9539480144 \times 10^{-6}(\text{eV m}^2)^{-1}$, and S_1 and S_2 are the spin operators for the two fermifish. At night, when the fermifish are resting and are thus in a state of lowest energy, he measures their total spin squared $(S_1 + S_2)^2$. He notices that he does not get the same result each night - sometimes he obtains 0, and sometimes he obtains $2\hbar^2$. Find the mass m of the fermifish in eV, rounded to the nearest eV.

14. Eugeniu the eel enthusiast has an uncharged metal sphere, which he places into his electric eel tank. In the absence of the sphere, the eels produce a uniform electric field $E = E_0 = 2.154180632827 \times 10^4$N/nC in the $\hat{z}$ direction. The electric field induces a non-uniform charge distribution on the sphere, which then distorts the electric field in the neighborhood of the sphere. Solve for the induced surface charge density on the sphere at $\theta = 0.42681131\pi$ radians and $\phi = 0.55721233\pi$ radians, where θ is the polar angle and ϕ is the azimuthal angle. Give your result in nC/cm^2, rounded to the nearest integer. Assume the permittivity of free space ϵ_0 is exactly 8.854×10^{-5}nF/cm.

15. Nick has a thermodynamic ensemble of $N \gg 1$ interesting jellyfish, each of whom can light up in $g+1$ ways: a single ground state with energy $E_0 = 0$ (i.e. not lit up), and $g \equiv e^{a\lambda}$ degenerate states all with energy λh corresponding to different colors of light shone at different intensities. Here, a is a dimensionless number equal to 19.547651403, $h = 2.1789601300 \times 10^{-18}$J, and λ is a dimensionless parameter. As a measure of the fraction of excited jellyfish, Nick computes f, the expected energy per particle normalized by the excited state energy
$$f(T, \lambda) = \frac{\langle E \rangle / N}{h\lambda}$$
As λ is taken to infinity, Nick notices that the graph of f as a function of temperature develops a discontunity at a particular critical temperature. Find this critical temperature in units of millikelvins. Use $k_B = 1.3806488000 \times 10^{-23}$J/K.

16. Ann is an oyster who has strung one of her pearls along a wire. Initially, the wire lies in the xz-plane, where z is the vertical direction. The shape of the wire at this time is $z = k|x|^3$, where $k = 118.9328997705\text{m}^{-2}$. Ann starts to rotate the wire around the z-axis at constant angular velocity $\omega = 7\text{rad/s}$ and notices that the pearl reaches a stable equilibrium at a nontrivial point along the wire. What is the distance from the z-axis to this special point in nanometers, rounded to the nearest nanometer? The gravitational field strength is $g = 9.8\text{m/s}^2$ exactly.



17. Seung-Yeop is a space-fish who is trying to use interferometry to detect the presence of gravitational waves. He believes that the metric he lives in is given by

$$ds^2 = -dt^2 + (1 + A\cos(\omega(t - z)))dx^2 + (1 - A\cos(\omega(t - z))dy^2 + dz^2$$

where $\omega = 46.8552838037\text{Hz}$ and $A \ll 1$ is a constant[1]. Seung-Yeop is sitting at the origin, and he decides that he will place two mirrors, one on the positive x-axis and one on the positive y-axis, at equal distance L from him. If he simultaneously emits two photons, one to each mirror, what is the smallest value of L that will result in the two signals arriving back to him at exactly the same time? Express your answer in meters, rounded to the nearest meter. Take $c \equiv 299792458\text{m/s}$.

18. Yuji has spontaneously tunneled into a fishy vacuum where the electroweak sector of the Standard Model is populated by 3042342 generations of leptons. Assuming that, in this vacuum, the weak angle is exactly given by $\sin^2(\theta_w) = 1/5$, that the fine structure constant at the W-boson mass is exactly $1/200$, and that the W boson mass is exactly 80 GeV, compute, at tree-level, the line-width for the positive polarization of the W boson to decay into any lepton pair in GeV. Assume that all the lepton masses are so small compared to m_W that they can be neglected.

19. Paul is a stargazing Pisces who is curious about the true size of the objects he is observing. He sees a galaxy which subtends an angle of exactly 0.00001 radians in the sky, and determines that it is at a redshift of 5.4337932157. Assuming Paul lives in a flat, matter-dominated Friedman-Robertson-Walker cosmology, what is the proper size of the galaxy? Express your

[1]For those who are curious, this is the metric of a +-polarized wave with angular frequency propagating in the z-direction in transverse-traceless gauge

answer in milli-Parsecs (mpc) rounded to the nearest milli-Parsec. Assume the Hubble constant today is exactly 70 km/s/Mpc and that the speed of light $c \equiv 299792458$m/s.

20. Ali the eel is swimming around in the background of two static, parallel NS5-branes in type IIB superstring theory. Of course, the supergravity solution describing this configuration, which is S-dual to an analogous black D5-brane system, is given as follows:

$$\begin{aligned} G_{\mu\nu} &= g\eta_{\mu\nu} \\ G_{ij} &= g^{-1}e^{2\Phi}\delta_{ij} \\ H_{ijk} &= -\epsilon_{ijk}{}^{l}\partial_l\Phi \\ e^{2\Phi} &= g^2 + \frac{Q_1}{2\pi^2(x^i - x_1^i)^2} + \frac{Q_2}{2\pi^2(x^i - x_2^i)^2} \end{aligned}$$

where greek indices refer to the parallel directions of the brane, latin indices refer to the transverse direction of the branes, Q_1 and Q_2 are the brane charges, x_1^i and x_2^i are the brane coordinates, G is the target space metric, and H is the Neveu-Schwarz 3-form. Suppose Ali is actually a D1-brane (i.e. the solitonic solution, not the fundamental string) stretched between the two NS5-branes. What is his string-frame mass in natural units (where $\hbar = c = 1$), rounded to the nearest integer? Set the Regge slope $\alpha' = 1/2$, the minimum brane separation $|x_2^i - x_1^i| = 11994.74072328$, and the string coupling $g = 10^{-7}$. Assume there are no excited gauge fields on the NS5-branes.

21. Jianyong is a fish floating on a $1+1$-d conformal field theory on the cylindrical boundary of a pure 2+1-d anti de-Sitter ocean. He wakes up one day and realizes that even though he has lived in the vacuum all his life, he is nonetheless very entangled with the rest of his $1+1$-d universe. He wishes to quantify this by computing the entanglement entropy of the interval in which he lives.

Fix Planck units with $\hbar = c = G_N = 1$.

Let R denote the radius of curvature of AdS, and consider pure 2+1-dimensional AdS global coordinates:

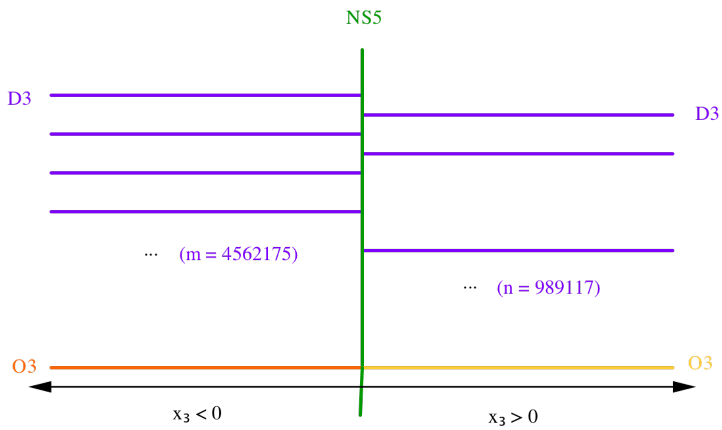
$$ds^2 = R^2(-\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\theta^2)$$

where $\rho = \infty$ is the boundary of AdS and the boundary is a cylinder whose spatial slices are circles parametrized by periodically-identified coordinate θ, $0 \leq \theta \leq 2\pi$.

Fix a time slice on the boundary and let A be the boundary interval $0 \leq \theta \leq \pi$. Fix the AdS radius R such that $\log(R) = 52991.4818246993$. To regulate divergences, take a hard cutoff, setting the boundary at $\rho = \rho_0$, where $\log(\rho_0) = 359236.211322482$. Compute the entanglement entropy S_A of A with the rest of the boundary universe and give its *logarithm*, $\log(S_A)$. You may use the Ryu-Takayanagi prescription.

22. Yu is a biophysicist. He is exploring a beach with a couple of colleagues. They find a tidepool of infinite extent containing an extremely interesting ecosystem. There is a D3-NS5 brane configuration in type II-B string theory, with coordinates $x_0, \ldots, x_9$:

- At $x_3 = x_7 = x_8 = x_9 = 0$ is a 5-d piece of Neverending Seaweed: an NS5 brane, parametrized by $x_0, x_1, x_2, x_4, x_5, x_6$.
- Attached to either side of the NS5 brane are semi-infinite parallel 3-d strands of Deaweed: D3 branes. There are $n = 989117$ D3 branes living at $x_3 > 0$ and $m = 4562175$ living at $x_3 < 0$. The D3-branes are parametrized by x_0, x_1, x_2, x_3 - i.e. each D3-brane lives at fixed $x_4, x_5, \dots, x_9$.
- Parallel to the deaweed, they find a piece of 3-d Orangeweed - that is, an O3 plane. They know it has discrete RR flux on one side of the NS5 brane, and both NS and RR flux on the other, but can't see which is which.



The figure depicts the configuration, where x_3 is the horizontal direction, the vertical direction schematically depicts x_4, x_5, x_6, the NS5 brane is depicted in green, the D3 branes in purple, and the O3 plane in orange, where the different colors reflect the different fluxes.

The three biophysicists instantly realize that the tidepool describes a bulk 4-d $\mathcal{N} = 4$ SYM gauge theory with a 3-d defect separating two regions with particular gauge groups, coupled to some 3-d bifundamental hypermultiplets that live on the defect.

Yu realizes that they can understand life in the whole tidepool just by studying what lives on the defect. He exclaims, "Up to Q-exact terms (where Q = supersymmetries), the system is entirely described by a weakly-coupled analytically-continued Chern-Simons ecosystem living on the defect!" He writes down the action

$$I = \frac{i\mathcal{K}}{4\pi} \int_{\text{defect}} \mathrm{Str} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) + (\text{Q-exact})$$

and specifies that $\mathcal{K} \in \mathbb{R}$, $\mathcal{A}$ can be thought of as a matrix-valued field acting on a certain $\mathbb{Z}_2$-graded vector space of dimension $m|n$ and Str represents a supertrace.

A second colleague too has found a way to study the bulk system this way, and immediately interrupts, "No, it's a strongly-coupled analytically-continued Chern-Simons ecosystem living on the defect, with a different gauge group!"

And finally, the third colleague replies, "Calm down, you are saying the same thing! The second Chern-Simons ecosystem is obtained by applying $S^{-1}TS$ to Yu's ecosystem," where S and T are S duality and T duality, respectively. Suppose Yu reported that the gauge group for the Chern-Simons theory was $\mathrm{OSp}(a|9124350)$, and the second colleague reported that the gauge group was $\mathrm{OSp}(c|d)$ where a, c, d are all positive integers. What is $a + c$?
