Conservation of Energy
Challenge Problem Solutions

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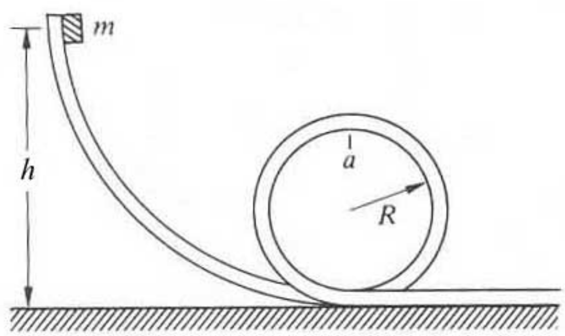
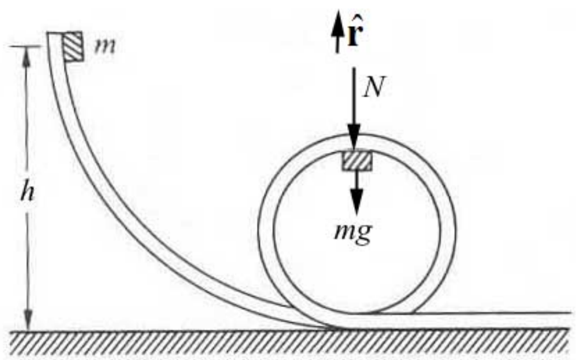
Problem 1

An object of mass m is released from rest at a height h above the surface of a table. The object slides along the inside of the loop-the-loop track consisting of a ramp and a circular loop of radius R shown in the figure. Assume that the track is frictionless. When the object is at the top of the track it pushes against the track with a force equal to three times it's weight. What height was the object dropped from?



Problem 1 Solution:

We choose polar coordinates with origin at the center of the loop. We choose the zero point for the potential energy $U = 0$ at the bottom of the loop.



Initial State: We choose for our initial state, the instant the mass is released. The initial kinetic energy $K_0 = 0$. The initial potential energy is non-zero, $U_0 = mgh$. So the initial mechanical energy is

$$E_0 = K_0 + U_0 = mgh.$$

Final State: We choose for our final state, the instant the mass is at the top of the loop-the-loop. The final kinetic energy $K_f = \frac{1}{2}mv_f^{\,2}$ since the mass is in motion at rest. The final potential energy is non-zero, $U_f = mg2R$. So the final mechanical energy is

$$E_f = K_f + U_f = 2mgR + \frac{1}{2}mv_f^{\,2}.$$

Non-conservative Work: Since we are assuming the track is frictionless, there is no non-conservative work.

Change in Mechanical Energy: The change in mechanical energy is therefore zero,

$$0 = W_{nc} = \Delta E_{mechanical} = E_f - E_0 .$$

Thus mechanical energy is conserved, $E_f = E_0$, or

$$2mgR + \frac{1}{2}mv_f^{\,2} = mgh.$$

Missing Condition: The normal force of the track on the object is perpendicular to the direction of the motion of the object so this force does zero work,

$$\vec{\mathbf{N}} \cdot d\vec{\mathbf{r}} = 0 .$$

Therefore the work-kinetic energy theorem does not account for the action of this force.

When there are forces that do no work in some direction, set up the Second Law in that direction,

$$\vec{\mathbf{F}}_{\perp} = m\vec{\mathbf{a}}_{\perp} .$$

We show the force diagram when the mass is at the top of the loop. Therefore Newton's Second Law in the radial direction, $\hat{\mathbf{r}}$, is

$$-mg - N = -mv_f^{\,2}/R .$$

Notice from the information given in the example, the normal force of the loop on the object is $\vec{\mathbf{N}} = -3mg\vec{\mathbf{r}}$. (This is the action-reaction pair).

Therefore the Second Law becomes

$$4mg = mv_f^{\,2} / R\,.$$

We can rewrite this condition in terms of the kinetic energy as

$$2mgR = \frac{1}{2} mv_f^{\,2}$$

Summary: Our two equations are therefore

$$2mgR + \frac{1}{2} mv_f^{\,2} = mgh$$
$$2mgR = \frac{1}{2} mv_f^{\,2}\,.$$

The second equation (from the Newton's Second Law) can be substituted into the conservation of mechanical energy equation to yield,

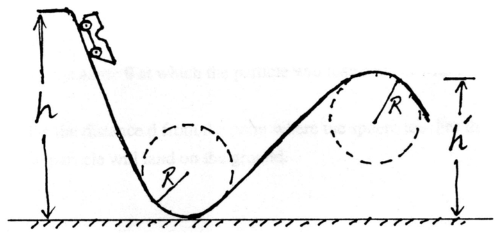
$$4mgR = mgh\,.$$

So the initial height is

$$h = 4R\,.$$

Problem 2

Consider a roller coaster in which cars start from rest at a height h_0, and roll down into a valley whose shape is circular with radius R, and then up a mountain whose top is also circular with radius R, as shown in the figure. Assume the contact between the car and the roller coaster is frictionless. The gravitational constant is g. (Note: the following comment was not in the original problem description.) Assume that the wheels of the car run inside a track which follows the path shown in the figure below, so the car is constrained to follow the track.



a) Find an expression the speed of the cars at the bottom of the valley.

b) If the net force on the passengers is equal to $8\ mg$ at the bottom of the valley, find an expression for the radius R of the arc of a circle that fits the bottom of the valley.

c) The top of the next mountain is an arc of a circle of the same radius R. If the normal force between the car and the track is zero at the top of the mountain, what is the height h_{top} of the mountain? (Note that this is a change from the original question, which asked for the value of h_{top} for which the car loses contact with the road at the top of the mountain. The problem has been changed because we realized that the car would have to be attached to the track to prevent it from flying off before it reached the top of the mountain. If the normal force is zero at the top, it would have to be negative before it reached the top.)

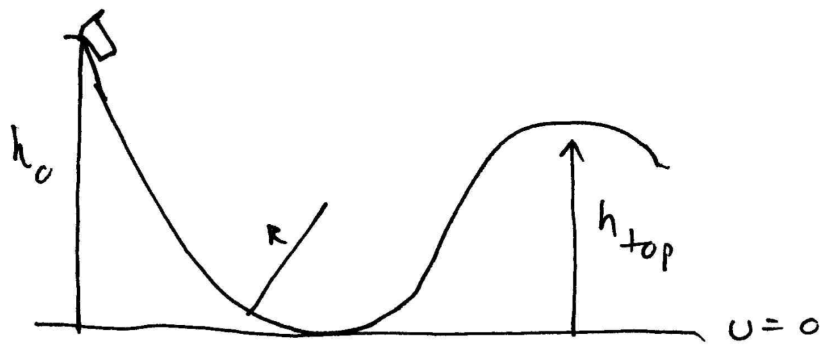
Problem 2 Solutions:

(a) Find an expression the speed of the car at the bottom of the valley.

Since we assume there is no loss of mechanical energy, the change in mechanical energy is zero,

$$0 = \Delta E_{mech} = \Delta U + \Delta K = (U_f - U_i) + (K_f - K_i) \qquad \textbf{(2.1)}$$

Set the potential energy at the ground to be zero.



Then

$$U_i = mgh_0 \,,\ K_i = 0$$
$$U_f = 0 \,,\ K_f = (1/2)mv^2_{bottom} \quad \textbf{(2.2)}$$

With these values Eq. (2.1) becomes

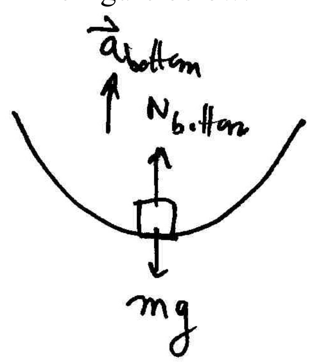
$$0 = (0 - mgh_0) + ((1/2)mv^2_{bottom} - 0) \quad \textbf{(2.3)}$$

Thus we can solve for the speed at the bottom

$$v_{bottom} = \sqrt{2gh_0} \quad \textbf{(2.4)}$$

(b) If the net force on the passengers is equal to 8 mg at the bottom of the valley, find an expression for the radius R of the arc of a circle that fits the bottom of the valley.

The free body diagram is shown in the figure below.



The force equation in the inward direction is then

$$N_{bottom} - mg = \frac{mv^2_{bottom}}{R} \quad \textbf{(2.5)}$$

If the net force on the passengers is equal to $8\ mg$, then

$$N_{bottom} - mg = 8mg \qquad \textbf{(2.6)}$$

Thus Eq. (2.5) becomes

$$8mg = \frac{mv_{bottom}^2}{R} \qquad \textbf{(2.7)}$$

From Eq. (2.4) we have that

$$\frac{mv_{bottom}^2}{R} = 2mgh_0 / R \qquad \textbf{(2.8)}$$

After Substituting Eq. (2.8) into Eq. (2.7), yields

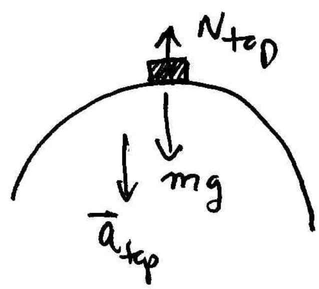
$$8mg = 2mgh_0 / R \qquad \textbf{(2.9)}$$

We can now solve for the radius of the circular trajectory at the bottom

$$R = h_0 / 4 \qquad \textbf{(2.10)}$$

(c) The top of the next mountain is an arc of a circle of the same radius R. If the car just loses contact with the road at the top of the mountain, what is the height h_{top} of the mountain?

At the top of the mountain the free body force diagram is shown in the figure below



Then the force equation becomes

$$mg - N_{top} = \frac{mv_{top}^2}{R} \qquad \textbf{(2.11)}$$

If the car just loses contact at the top, then $N_{top} = 0$, and Eq. (2.11) becomes

$$mg = \frac{mv_{top}^2}{R} \qquad \textbf{(2.12)}$$

We can use as our initial state, the original height and the final state the top of the mountain for our energy equation. Then we have that

$$\begin{aligned} U_i &= mgh_0 \ , \ K_i = 0 \\ U_f &= mgh_{top} \ , \ K_f = (1/2)mv_{top}^2 \end{aligned} \qquad \textbf{(2.13)}$$

Eq. (2.1) becomes

$$0 = (mgh_{top} - mgh_0) + (1/2)mv_{top}^2 \qquad \textbf{(2.14)}$$

Thus we can rewrite Eq. (2.14) after dividing through by R as

$$\frac{mv_{top}^2}{R} = \frac{2mg(h_0 - h_{top})}{R} \qquad \textbf{(2.15)}$$

Substituting Eq. (2.15) into Eq. (2.12) yields

$$mg = \frac{2mg(h_0 - h_{top})}{R} \qquad \textbf{(2.16)}$$

We can solve Eq. (2.16) for the h_{top}

$$h_{top} = h_0 - \frac{R}{2} \qquad \textbf{(2.17)}$$

Now substitute Eq. (2.10) and find that

$$h_{top} = h_0 - \frac{h_0}{8} = \frac{7h_0}{8} \qquad \textbf{(2.18)}$$

Problem 3

Find the escape speed of a rocket from the moon. Ignore the rotational motion of the moon. The mass of the moon is $m = 7.36\times10^{22}\,kg$. The radius of the moon is $R = 1.74\times10^{6}\,m$. The universal gravitation constant $G = 6.7\times10^{-11}\,N\cdot m^{2}\cdot kg^{-2}$.

Problem 3 Solution:

The "escape velocity" (really the escape *speed*) is the magnitude of the velocity for which the net mechanical energy, the sum of the kinetic energy and the gravitational potential energy, is zero, where the gravitational potential energy is defined so that the potential energy goes to zero in the limit of an infinite distance. This means that at any finite distance the potential energy is negative, and hence the kinetic energy is positive, and the object is still moving with nonzero speed.

We have then that

$$\frac{1}{2}mv_{\text{esc}}^{2} - G\frac{m\,m_{moon}}{R} = 0, \tag{3.1}$$

which is readily solved for

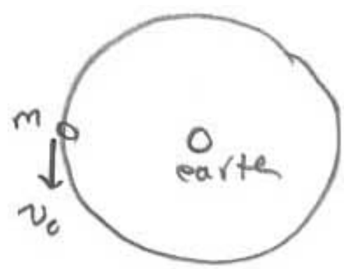
$$\begin{aligned} v_{\text{esc}} &= \sqrt{\frac{2G\,m_{\text{moon}}}{R}} \\ &= \sqrt{\frac{2\left(6.674\times10^{-11}\,\text{m}^{3}\cdot\text{kg}^{-1}\cdot\text{s}^{-2}\right)\left(7.36\times10^{22}\,\text{kg}\right)}{1.74\times10^{6}\,\text{m}}} = 2.38\times10^{3}\,\text{m}\cdot\text{s}^{-1}. \end{aligned} \tag{3.2}$$

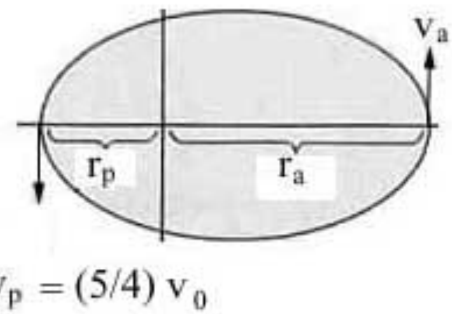
Problem 4

A satellite of mass $m = 5.0 \times 10^2\ kg$ is initially in a circular orbit about the earth with a radius $r_0 = 4.1 \times 10^7\ m$ and velocity $v_0 = 3.1 \times 10^3\ m/s$ around the earth. Assume the earth has mass $m_e = 6.0 \times 10^{24}\ kg$ and radius $r_e = 6.4 \times 10^6\ m$. Since $m_e >> m$, you may find it convenient to ignore certain terms. Justify any terms you choose to ignore.

a) What is the magnitude of the gravitational force acting on the satellite? What is the centripetal acceleration of the satellite?

b) What are the kinetic and potential energies of the satellite earth system? State any assumptions that you make. Specify your reference point for zero potential energy. What is the total energy?





As a result of a satellite maneuver, the satellite trajectory is changed to an elliptical orbit. This is accomplished by firing a rocket for a short time interval and increasing the tangential speed of the satellite to $(5/4)v_0$. You may assume that during the firing, the satellite does not noticeably change the distance from the center of the earth.

c) What are the kinetic and potential energies of the earth-satellite at the point of closest approach? Specify your reference point for zero potential energy. What is the total energy?

d) How much energy was necessary to change the orbit of the satellite?

e) What speed would the satellite need to acquire so that it can escape to infinity?

Problem 4 Solutions:

a) The magnitude of the gravitational force is $\left|\vec{F}\right| = \frac{Gm_e m_s}{r_0^2} = 117N$. The centripetal acceleration is $\vec{a} = -\frac{v_0^2}{r_0}\hat{r}$ and $\frac{v_0^2}{r_0} = 0.23m \cdot s^{-2}$

b) We will suppose the earth as fixed and we will choose the 0 point for the potential energy to be ∞. Then we have $K = \frac{1}{2}m_s v_0^2 = 24 \times 10^8 J$ and $U = -\frac{Gm_e m_s}{r_0} = -2K = -48 \times 10^8 J$

c) At the point of closest approach that is just after the rocket goes off we have $K = \frac{25}{32}m_s v_0^2 = 37.5 \times 10^8 J$ while $U = -\frac{Gm_s m_e}{r_0} = -48 \times 10^8 J$ adding these we get for the total energy $E = -\frac{7}{32}m_s v_0^2 = -10.5 \times 10^8 J$

d) The total energy of the system after the rocket went off minus the total energy of the system before it went on is the total work done by the rocket that is: $W = \frac{9}{32}m_s v_0^2 = 13.5 \times 10^8 M$

e) For escaping to ∞ the satellite must have total energy 0, that is $v = \sqrt{2}v_0 = 4.4 \times 1-^3 \ m \cdot s^{-1}$ with this speed, in fact, $K = m^s v_0^2 = -U$

Problem 5

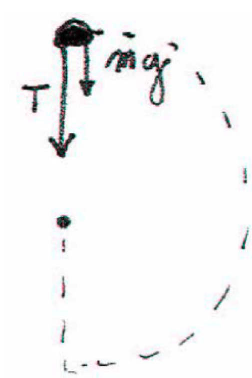
A ball of negligible radius is tied to a string of radius R. A man whirls the string and stone in a vertical circle. Assume that any non-conservative forces have negligible effect. Show that if the string is to remain taut at the top of the circle, the speed at the bottom of the circle must be at least $\sqrt{5gR}$.

Problem 5 Solution:

Let the initial state correspond to the stone at the bottom of the circle. Take the zero point for gravitational potential energy to be at the bottom of the circle where the speed of the ball is denoted by v_b. Then initial mechanical energy is then $E_0 = mv_b^2/2$ and the final mechanical energy, is $E_f = mv_t^2/2 + 2mgl$ where the speed of the ball at the top of the circle is denoted by v_t. Equating these energies (there are no nonconservative forces) yields

$$mv_b^2/2 = mv_t^2/2 + 2mgR. \quad \textbf{(5.1)}$$

When the string is vertical, the ball will be accelerating inward, and the gravitational force and the tension force are inward.



Therefore Newton's Second Law

$$T + mg = m\frac{v_t^2}{R}. \quad \textbf{(5.2)}$$

As the speed at the top decreases eventually the tension becomes zero and the string is no longer taut. So the constraint condition is the minimum speed at the top occurs when

$$T = 0. \quad \textbf{(5.3)}$$

At this minimum speed, Eq. (5.2) becomes

$$mg = m\frac{v_{t,\min}^2}{R}. \tag{5.4}$$

or

$$mgR = mv_{t,\min}^2. \tag{5.5}$$

We can now substitute Eq. (5.5) into Eq. (5.1) to find that

$$mv_{b,\min}^2 / 2 = mgR / 2 + 2mgR. \tag{5.6}$$

We now solve for the minimum speed at the bottom of the circle necessary such that the string just remains taut at the top

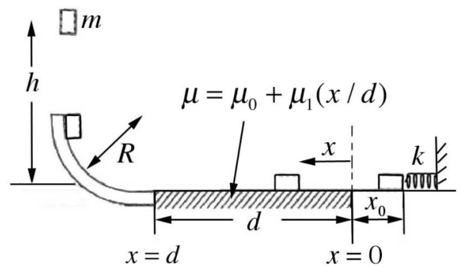
$$v_{b,\min} = \sqrt{5gR}. \tag{5.7}$$

Problem 6:

An object of mass m is released from an initial state of rest from a spring of constant k that has been compressed a distance x_0. After leaving the spring (at the position $x = 0$ when the spring is unstretched) the object travels a distance d along a horizontal track that has a coefficient of friction that varies with position as

$$\mu = \mu_0 + \mu_1(x / d).$$

Following the horizontal track, the object enters a quarter turn of a frictionless loop whose radius is R. Finally, after exiting the quarter turn of the loop the object travels vertically upward to a maximum height, h, (as measured from the horizontal surface). Let g be the gravitational constant. Find the maximum height, h, that the object attains. Express all answers in terms of m, k, x_0, g, μ_0, μ_1, d and R; not all variables may be needed.



Problem 6 Solution:

This problem may seem complicated at first (all those parameters!), but the work-energy theorem makes it tractable, even simple. Take the initial state to be when the spring is compressed the distance x_0 and the final state to be when the object is at its maximum height. The initial energy is then $E_{\text{initial}} = \dfrac{1}{2} k x_0^2$ and the final energy is $E_{\text{final}} = mgh$; neither expression contains a kinetic energy term.

The nonconservative work is that done by friction. The magnitude of the friction force is $f = \mu mg$ and the nonconservative work is

$$W_{\text{nc}} = -\int f\, dx = -mg \int \mu\, dx. \qquad \textbf{(6.1)}$$

If the coefficient of friction were constant ($\mu_1 = 0$), this wouldn't be a difficult problem at all, and with the expression given, the integral is not at all hard. We have

$$
\begin{aligned}
W_{\mathrm{nc}} &= -mg\int \mu\, dx \\
&= -mg\int_0^d \left(\mu_0 + \mu_1 \left(x/d\right)\right)dx \\
&= -mg\left(\mu_0 d + \mu_1 d/2\right) \\
&= -mgd\left(\mu_0 + \mu_1/2\right).
\end{aligned} \qquad \textbf{(6.2)}
$$

It should be noted that in the figure, x increases from right to left. In any event, the non-conservative work done by friction is negative.

The work-energy theorem is then

$$
mgh - \frac{1}{2}k\,x_0^2 = -mgd\left(\mu_0 + \mu_1/2\right), \qquad (6.3)
$$

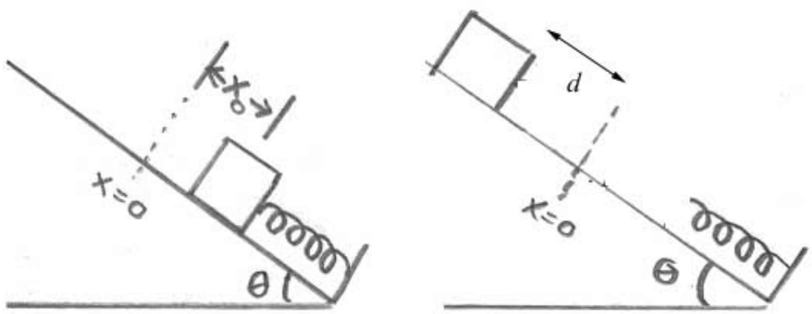
which is easily solved for

$$
h = \frac{k\,x_0^2}{2mg} - d\left(\mu_0 + \mu_1/2\right). \qquad (6.4)
$$

The result of Equation (6.4) must be qualified, in that if that result were negative, the friction would be enough to stop the object before it entered the loop. Also, there is no reason why μ_1 could not be negative, which would mean the friction force decreases in magnitude with increasing x. However, if this were the case, we would have to have $\mu_0 + \mu_1 \geq 0$.

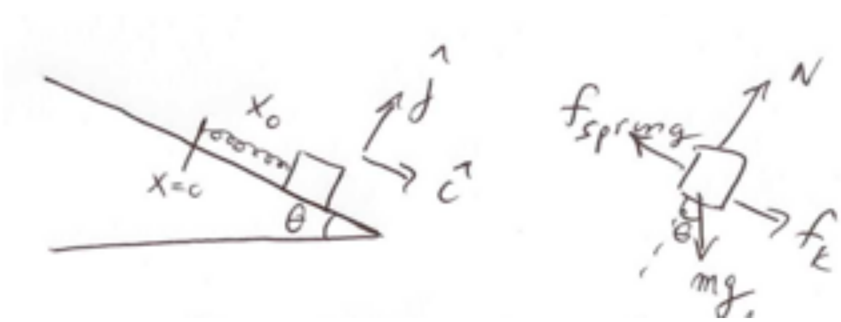
Problem 7

A object of mass m is pushed against a spring at the bottom of a plane that is inclined at an angle θ with respect to the horizontal and held in place with a catch. The spring compresses a distance x_0 and has spring constant k. The catch is released and the object slides up the inclined plane. At x = 0 the object detaches from the spring and continues to slide up the inclined plane.



Assume that the incline plane has a coefficient of kinetic friction μ_k. How far up the inclined plane does the object move from the point where the object detaches from the spring?

Problem 7 Solution:



Stage 1: The spring is released and the object moves back to equilibrium.

$$\Delta K = \int \bar{F}_{total} \cdot dr$$

$$\frac{1}{2}mv_p^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x=0} -Kx\,dx + \int_{x_0}^{x=0} \mu_k N\,dx + \int_{x_0}^{x=0} mg\sin\theta\,dx$$

$$\sin\theta\left(\bar{F}_{total}\right)_x = -kx\hat{i} + \mu_k N\hat{i} + mg\sin\theta\hat{i}$$

$$\text{note that } F_y = ma_y = 0$$

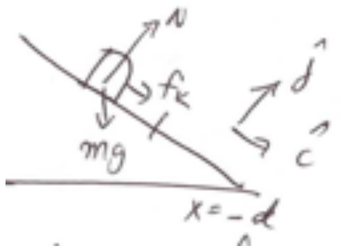
$$N - mg\cos\theta = 0 \quad or \quad N = mg\cos\theta$$

the object is released at rest $s = v_0 = 0$

thus

$$\frac{1}{2}mv_1^2 = -\int_{x_0}^{x=0} kx\,dx + \int_{x_0}^{x=0} u_k mg\cos\theta\,dx + \int_{x_0}^{x=0} mg\sin\theta\,dx$$

Stage 2: object slides up inclined plane until it comes to rest.

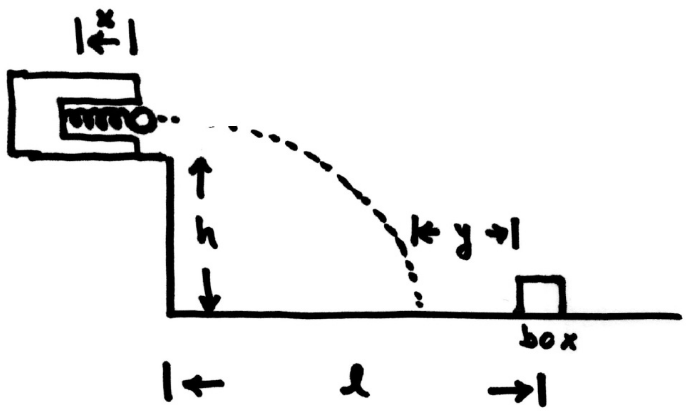


$$\Delta K = \int_{x=0}^{x=-d} \vec{F}\cdot d\vec{r}$$

$$\Delta K = +\int_{0}^{x=-d} \mu_k mg\cos\theta\,d\theta + \int mg\sin\theta\,d\theta$$

Problem 8

Two children are playing a game, which they try to hit a small box using a spring -loaded marble gun, which is fixed rigidly to a table and projects a marble of mass m horizontally from the edge of the table. The edge of the table is a height h above the top of the box. The spring has a spring constant k and the edge of the box is some unknown horizontal distance l away from the table. The first child compresses the spring a distance x and finds that the marble falls short of its target by a horizontal distance y. How far should the second child compress the spring in order to land in the box? Let g denote the gravitational acceleration. Express your answer in terms of k, m, x, g, h, and y as needed but do not use the unknown distance l.



Problem 8 Solution:

We first apply work-kinetic energy to determine the kinetic energy of the ball after it has been released from the spring. We choose a coordinate system with origin at the position of the ball when the spring is at rest in the equilibrium position. We choose the $\hat{\mathbf{i}}$ unit vector to point in the direction the ball the spring is compressed. We choose the coordinate function $x(t)$ to denote the position of the body with respect to the origin (equilibrium position) at time t. The spring force on the body is given by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}} \tag{8.1}$$

The work done by the spring on the ball is just the area under the curve for the interval $x_0 = \mathrm{x}$ (the amount the spring was compressed) to $x_f = 0$,

$$W = \int_{x=x_0}^{x_f=0} F_x dx = \int_{x=x_0}^{x_f=0} (-kx)dx \tag{8.2}$$

This integral is straightforward and the work done by the spring force on the body is

$$W = \int_{x=x_0 0}^{x_f=0} (-kx)dx = \frac{1}{2}kx^2 > 0. \quad \textbf{(8.3)}$$

So the work-kinetic energy theorem $W_{\text{total}} = \Delta K$ becomes

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2_{\text{exit}}. \quad \textbf{(8.4)}$$

We can solve for the exit velocity

$$v_{\text{exit}} = \sqrt{\frac{k}{m}}x. \quad \textbf{(8.5)}$$

Now choose an origin at the exit point, with positive y-axis pointing up and positive x-axis pointing in the direction of the exit velocity. The equations of projectile motion for the horizontal and vertical motion are

$$x = v_{\text{exit}}t, \quad \textbf{(8.6)}$$

$$y = -\frac{1}{2}gt^2. \quad \textbf{(8.7)}$$

The ball hits the ground a horizontal distance $x_f = l - y$ with a vertical drop $y_f = -h$. Thus the vertical equations of motion when the ball hits the ground is

$$-h = -\frac{1}{2}gt_f^{\,2}. \quad \textbf{(8.8)}$$

We can solve this equation for the time of flight,

$$t_f = \sqrt{\frac{2h}{g}}. \quad \textbf{(8.9)}$$

The horizontal equation becomes

$$l - y = v_{\text{exit}}t_f = \sqrt{\frac{k}{m}}xt_f = \sqrt{\frac{2hk}{gm}}x, \quad \textbf{(8.10)}$$