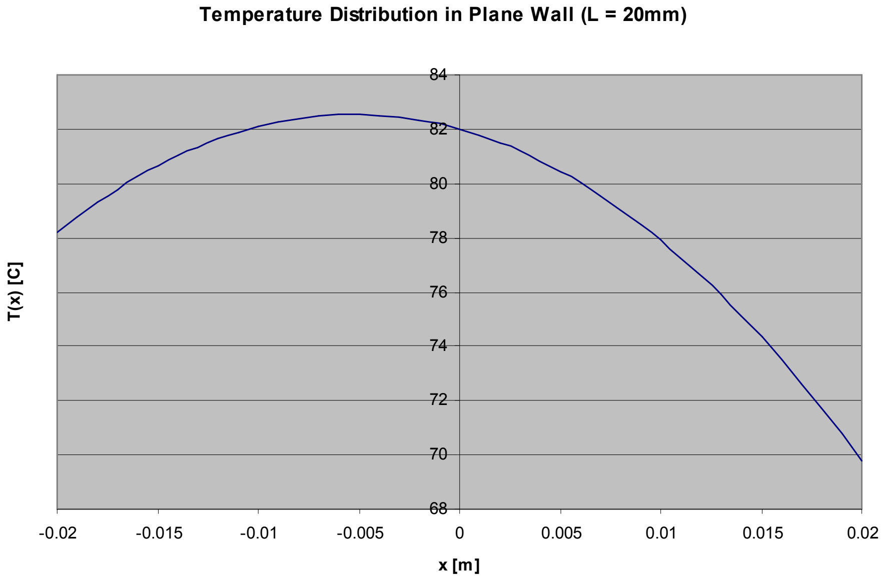
2.005 Thermal Fluids Engineering I **Fall`08**
Problem Set 7 Solutions

Problem 1

a)


Temperature Distribution in Plane Wall (L = 20mm)

b) The heat diffusion equation (eq. 6.52 in the course reader) is: $\rho c \dfrac{\partial T}{\partial t} = \nabla\left(k \nabla T\right) + \rho \dot{q}_{gen}$

We use it with the following conditions:

Steady state: $\dfrac{\partial T}{\partial t} = 0$

One-dimensional: $\dfrac{\partial T}{\partial y} = \dfrac{\partial T}{\partial z} = 0$

Volumetric heat generation *q'''*: $\rho \dot{q}_{gen} = q'''$

Constant k

which gives the differential equation we'd have to solve:

$$0 = k\frac{d^2T}{dx^2} + q'''$$

…with boundary conditions:

Convection heat flux at x = −L: $k\left.\dfrac{dT}{dx}\right|_{x=-L} = h_{-L}\left(T(-L) - T_\infty\right)$

Convection heat flux at x = L: $k\left.\dfrac{dT}{dx}\right|_{x=L} = -h_L\left(T(L) - T_\infty\right)$

where h_{-L} and h_L are the convection heat transfer coefficients at $x = -L$ and $x = L$, respectively. Note the signs, arising from the direction of heat transfer at each surface (remember, $q = -k(dT/dx)$!)

c) There are at least two ways of doing this. One is to use the temperature distribution function with the differential equation to find:

$$0 = k\frac{d^2}{dx^2}\left(A + Bx + Cx^2\right) + q'''$$

$$0 = k\left(2C\right) + q'''$$

$$-q''' = 2kC = 4\cdot(5\ \text{W/mK})\cdot\left(-2\times10^{4}\ ^\circ\text{C/m}\right)$$

$$\therefore q''' = 2\times10^{5}\ \text{W/m}^3$$

Another equally valid way is to draw the system boundary around the wall, apply the First Law, and say that the total heat generation is equal to the heat transfer rate out of the wall (*k*A*dT*/*dx* at the surfaces). This gives the same answer as above.

d) We know that: $q_x = -k\dfrac{dT}{dx}$

so, we just use the given function for *T*:

$$q_x = -k\frac{d}{dx}\left(A + Bx + Cx^2\right) = -k(B + 2Cx)$$

Plugging in *x* = −*L* and *x* = *L* gives the values of the two heat fluxes:

$$q_x(-L) = -k\left(B + 2C(-L)\right) = -2950\ \text{W/m}^2$$

or, +2950 W/m^2 *out of the wall*

$$q_x(L) = -k\left(B + 2C(L)\right) = 5050\ \text{W/m}^2$$

or, +5050 W/m^2 *out of the wall*

e) Here, we just use heat flux *out of the wall* and the temperature of the wall and the ambient to find the heat transfer coefficient, since we know that for convection:

$$q = h(T_s - T_\infty)$$

(note, of course, that *q* is heat transfer *per unit area*).

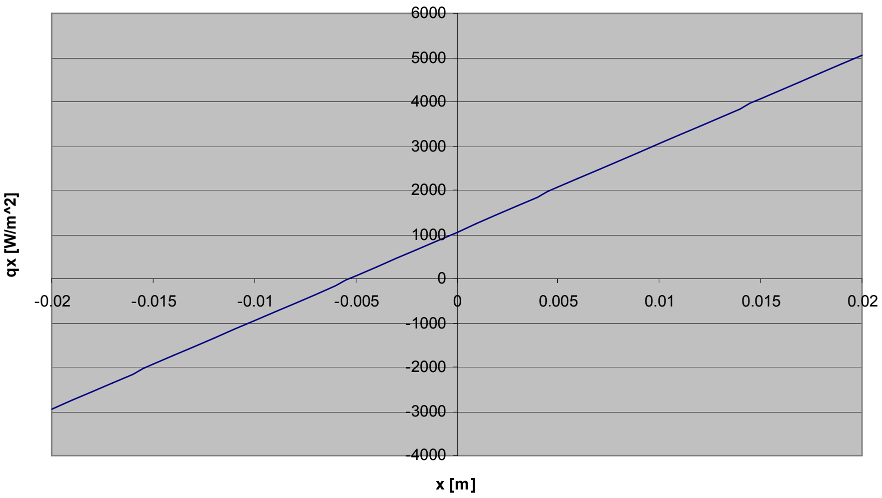
Computing the surface temperatures T_s at *x* = −*L* and *x* = *L* using the given temperature distribution (which are 78.2 C and 69.8 C respectively), and using the surface heat flux values from part d), we find:

$$h_{-L} = \frac{\left(2950\ \text{W/m}^2\right)}{(78.2\ ^\circ\text{C} - 20\ ^\circ\text{C})} = 50.7\ \text{W/m}^2\text{K}$$

$$h_{L} = \frac{\left(5050\ \text{W/m}^2\right)}{(69.8\ ^\circ\text{C} - 20\ ^\circ\text{C})} = 101\ \text{W/m}^2\text{K}$$

f) We already calculated the heat flux as a function of position in part (d).

$$q_x = -k(B + 2Cx)$$



Yes, the heat flux is zero at about x ~ −0.005 m. Also, the heat flux distribution is a linear function of position.

g) There's a nice conceptual argument here: The system was at steady state, with the total heat flux *out* equal to the total heat generation per unit area of the wall ($q'''L$). Immediately after the heat source is switched off, the heat flux *out* will still be equal to this value, since the temperature profile (and most importantly, dT/dx at the walls) hasn't changed yet! So, the rate of change is just $-(q'''L) = -8000\ \mathrm{W/m^2}$. Since the thickness of the wall is 40 mm, the rate of change of energy is $-8000/0.04 = -2\times10^5\ \mathrm{W/m^3}$ on a volume basis.

Alternatively, we could just use the heat fluxes out of either side of the walls to get the same result (again remember that the temperature profile won't have had time to change yet, and so all the heat fluxes will be the same immediately after the heat source is switched off):

$$-(2950\ \mathrm{W/m^2} + 5050\ \mathrm{W/m^2}) = -8000\ \mathrm{W/m^2}$$

Yet another way of doing it is to substitute the temperature profile into the differential equation, and set q''' to zero:

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + 0 = k\frac{\partial^2}{\partial x^2}\left(A + Bx + Cx^2\right) = 2kC$$

…and then combine with the time derivative of $\Delta E = mc\Delta T$, per unit area. Note, the mass of the wall per unit area is just its density times its thickness.

$$\frac{1}{Area}\frac{\partial E}{\partial t} = \frac{m}{Area}c\frac{\partial T}{\partial t} = \rho(2L)c\left(\frac{2kC}{\rho c}\right) = 4LkC = -8000\ \mathrm{W/m^2}$$

h) The wall will eventually reach the temperature of its environment, T = 20 °C.

To find the total energy that must be removed from the wall per unit area, we have to integrate $dE = (dm)c\Delta T$ across the width of the wall, noting that $(dm / Area)$ is equal to density times dx:

$$\int_{-L}^{L}\frac{dE}{Area} = \int_{-L}^{L}\frac{dm}{Area}c\left(T(x) - T_\infty\right)$$

$$\frac{\Delta E}{Area} = \int_{-L}^{L}\rho c\left(T(x) - T_\infty\right)dx = 2\rho c\int_{-L}^{L}\left(T(x) - T_\infty\right)dx$$

$$= 2\rho c\int_{-L}^{L}\left(a + bx + cx^2 - T_\infty\right)dx = 2\rho c\left(ax + b\frac{x^2}{2} + c\frac{x^3}{3} - T_\infty x\right)_{x=-L}^{x=L}$$

$$= 2\times2600\times800\left(82\times0.04 - 210\times0 - \frac{20000}{3}\left(0.02^3 - (-0.02)^3\right) - 20\times0.04\right)$$

$$= 4.94\times10^6\ J/m^2$$

Problem 2

(a) To justify the use of a lumped capacitance analysis, we need to determine the Biot ($Bi = hL_c/K$) number.

Chip: For the chip, we will use $L_c = t_{ch}/2 = 1$ mm, since there is heat transfer at all surfaces; at worst, heat has to diffuse across only half the chip thickness.

$$Biot = \frac{hL_C}{k} = \frac{h}{k}\frac{t_{ch}}{2} = \frac{50[W/m^2K]}{150[W/mK]}\times\frac{0.002[m]}{2} = 0.333\times10^{-3} \ll 1$$

Solder: For the solder, we will use $L_c = V/A_s$. (You can also use the radius or the diameter as L_c).

$$Biot = \frac{hL_C}{k} = \frac{hV/A_S}{k} = \frac{h}{k} \times \frac{4/3 \times \pi r^3}{4\pi r^2} = \frac{h}{k} \times \frac{D}{6} = \frac{50[W/m^2K]}{80[W/mK]} \times \frac{0.002[m]}{6} = 0.20833 \times 10^{-3} \langle\langle 1$$

Substrate: For the substrate, we will use $L_c = t_{sb}$ = 10 mm, since heat has to diffuse all the way to the bottom.

$$Biot = \frac{hL_C}{k} = \frac{h}{k} t_{sb} = \frac{50[W/m^2K]}{40[W/mK]} \times 0.010[m] = 0.0125 \langle\langle 1$$

Since the Biot number for each component is much less than 1, lumped capacitance analysis is valid for each component.

(b) From the application of the 1st Law to a lumped solid with convection heat transfer,

$$-hA_S(T-T_\infty) = \rho V c \frac{dT}{dt},$$

where A_s is the surface area of the solid that is in contact with the fluid.

We now define $\theta \equiv T - T_\infty$. Recognizing that $\frac{d\theta}{dt} = \frac{dT}{dt}$, the differential equation becomes

$$-\theta = \frac{\rho V c}{hA_S} \frac{d\theta}{dt}$$

Integrating the above equation from t = 0 to t = t, we get $t = \frac{\rho V c}{hA_S} \ln \frac{\theta_i}{\theta}$ where $\theta_i \equiv T_i - T_\infty$

$$\Rightarrow \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_S}{\rho V c}\right) t\right]$$

The thermal time constant is defined as $\tau_t = \frac{\rho V c}{hA_S} (= R_t C_t)$.

The above analysis is valid for all three components – chip, solder and substrate. Only the numbers change.

Chip:

$$\tau_t = \frac{\rho V c}{hA_S} = \frac{\rho c}{h} \times \frac{tL}{4t + 2L} = \frac{2300[Kg/m^3] \times 710[J/kgK]}{50[W/m^2K]} \times \frac{0.002[m] \times 0.015[m]}{4 \times 0.002[m] + 2 \times 0.015[m]} = 25.8s$$

Solder:

$$\tau_t = \frac{\rho V c}{hA_S} = \frac{\rho c}{h} \times \frac{D}{6} = \frac{11000[Kg/m^3] \times 130[J/kgK]}{50[W/m^2K]} \times \frac{0.002[m]}{6} = 9.5s$$

Substrate:

$$\tau_t = \frac{\rho V c}{hA_S} = \frac{\rho c}{h} \times \frac{tL}{2L + 4t} = \frac{4000[Kg/m^3] \times 770[J/kgK]}{50[W/m^2K]} \times \frac{0.010[m] \times 0.025[m]}{4 \times 0.010[m] + 2 \times 0.025[m]} = 616.0s$$

(c) From the previous part $\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_S}{\rho V c}\right) t\right]$,

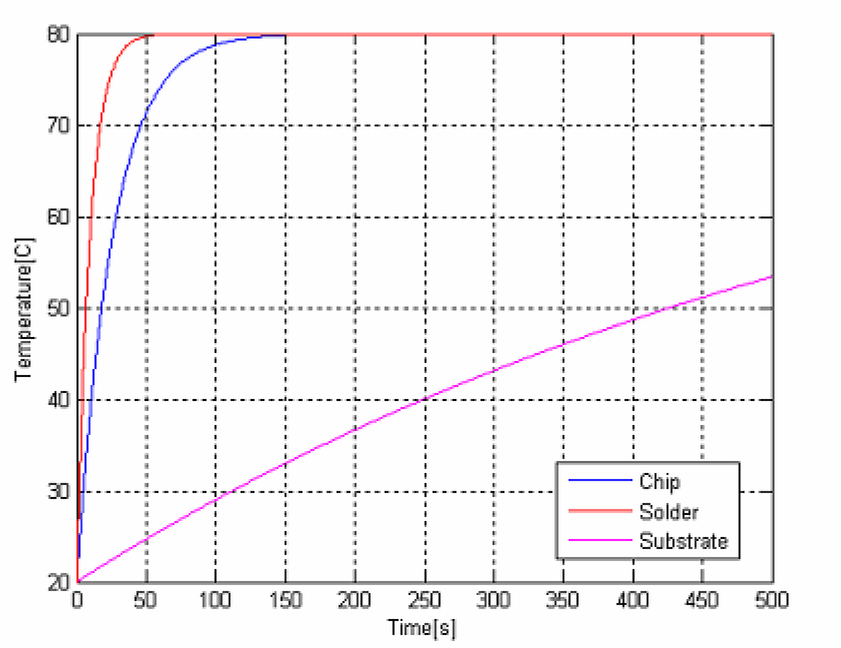
Therefore, $T(t) = T_\infty + (T_i - T_\infty)\exp[-t/\tau_t]$ where $T_i = 20\,^oC$ and $T_\infty = 80\,^oC$

For the three components, the temperature histories are:

Chip: T(t) = 80-60exp(-t/25.8)

Solder: T(t) = 80-60exp(-t/9.5)

Substrate: T(t) = 80-60exp(-t/616.0)

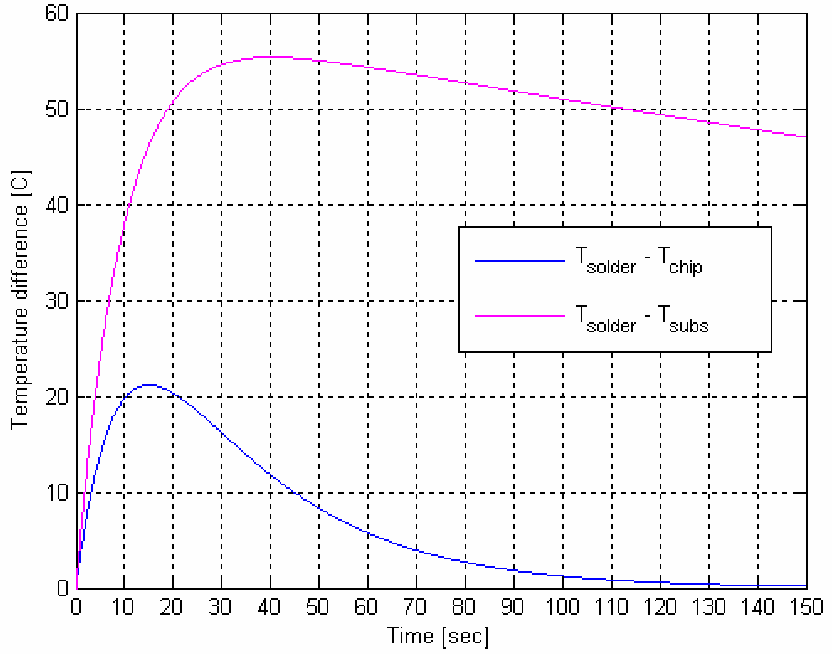


(d) Recognizing that $(T - T_i)/(T_\infty - T_i) = 1 - (\theta/\theta_i)$,
$(T - Ti)/(T_\infty - T_i) = 0.99$ yields $\theta/\theta_i = 0.01$, or $\theta i/\theta = 100$.

Since $t = \frac{\rho V c}{h A_S} \ln \frac{\theta_i}{\theta}$, it follows that the time required for a component to experience 99% of its maximum possible temperature rise is $t = \tau_t \ln(100) = 4.605\tau_t$

The times taken to experience 99% of the temperature rises are:
Chip: $t_{99} = 4.605\,\tau_t = 118.8$ s
Solder: $t_{99} = 43.8$ s
Substrate: $t_{99} = 2837.8$ s

(e) Histories of the temperature differences between a solder ball and its adjoining components are shown below.



Commensurate with their time constants, the fastest and slowest responses to heating are associated with the solder and substrate, respectively. Accordingly, the largest temperature difference is between these two components, and it achieves a maximum value of 55.34°C at 40.4 seconds.

(f) With the 4-fold increase in heat transfer coefficient, up to h=200 W/m^{2}K, associated with the use of a dielectric liquid to heat the components, the time constants ($\tau_t = \rho Vc / hA_s$) are all reduced by a factor of 4

Chip: τ_t = 6.45s

Solder: τ_t = 2.75s

Substrate: τ_t = 154s

In turn, the times required to achieve 99% of the maximum temperature rise ($t_{99} = \tau_t \ln(\theta_i / \theta) = 4.605\,\tau_t$) are reduced by a factor of 4 as well.
Chip: t_{99} = 29.7s
Solder: t_{99} = 10.95s
Substrate: t_{99} = 709.45 s

Therefore, the time savings is approximately 75%.

Problem 3
a) Show temperature gradients are in the glass fiber, not the copper balls.
We must consider what the Biot number actually tells us about a heat transfer process. It is a comparison of internal resistance to external resistance.

$$Bi = \frac{R_{\text{int}}}{R_{ext}} = \frac{\left(\frac{l}{kA}\right)_{Cu}}{\left(\frac{l}{kA}\right)_{gf}}$$

We have flexibility to choose our lengths and areas for this equation. Remember that Biot numbers are used for order of magnitude calculations. Let us assume that the area for conduction through the copper ball is the same as the area of the glass fibre (the area for conduction through the copper is actually much larger and this approximation will only overestimate the Biot number). We will choose the diameter of the copper ball as its characteristic length.
So l_{Cu} = 0.02 m, $A_{Cu} = A_{gf}$. The length of the glass fibre is l_{gf} = 0.01 m.
The thermal conductivities are k_{Cu} = 400 W/mK, k_{gf} = 1 W/mK

So the Biot number is $$Bi = \frac{\frac{0.02}{400(A_{Cu})}}{\frac{0.01}{1(A_{gf})}} = 0.005$$

Remember that this is an overestimation of the Biot number – the actual value is expected to be much lower than this. Even so, it is much less than unity, which means that the resistance to conduction through the glass fiber is much greater than the resistance to conduction through the balls. Therefore, we can say that the temperature gradients in the fiber dominate this process.

b) Show $T_1(t) + T_2(t)$ = constant for the process. Find T_1(final) & T_2(final).
Write the First Law for a system consisting of the balls and the glass fiber. There is no work transfer from the system. Since the fine wire has negligible thermal conductance, there is no heat transfer to the system. Therefore the First Law reduces to $\frac{dE}{dt} = \dot{Q} - \dot{W} = 0$

Remember that the glass fiber has no heat capacity.

$$\Rightarrow \frac{dE}{dt} = m_{Cu,1} c_{Cu,1} \frac{dT_1}{dt} + m_{Cu,2} c_{Cu,2} \frac{dT_2}{dt} = 0$$

Since the masses and specific heats of both balls are the same, $\frac{d(T_1+T_2)}{dt}=0$

The sum of the temperatures does not change with time.
Therefore, $T_1(t) + T_2(t) = T_1(t=0) + T_2(t=0) = 425 + 325 = 750$.
Since both balls reach the same final temperature, $T_1(\text{final}) = T_2(\text{final}) = T(\text{final}) = 750/2 = 375$ K

c) Develop a differential eqn. for temperature of ball 1 as a function of time and temperature difference.

Use the First Law for ball 1 only (no work transfer) $\frac{dE}{dt}=\dot{Q}$

The heat transfer to ball 1 is $\dot{Q}=-\frac{k_{gf}A_{gf}}{l_{gf}}\left(T_1-T_2\right)$

Using the energy constitutive relation for ball 1, we get the following differential equation:

$$m_{Cu,1}c_{Cu,1}\frac{dT_1}{dt}=-\frac{k_{gf}A_{gf}}{l_{gf}}\left(T_1-T_2\right) \qquad (1)$$

d) Find the time for ball 1 to reach 390K.

The corresponding differential equation for ball 2 is $m_{Cu,2}c_{Cu,2}\frac{dT_2}{dt}=-\frac{k_{gf}A_{gf}}{l_{gf}}\left(T_2-T_1\right)$ (2)

Subtracting (2) from (1) and noting that the masses and heat capacities are equal, we get

$$m_{Cu,1}c_{Cu,1}\frac{d\left(T_1-T_2\right)}{dt}=-2\frac{k_{gf}A_{gf}}{l_{gf}}\left(T_1-T_2\right)$$

Let $\theta = T_1 - T_2$

$$\frac{d\theta}{dt}=-2\frac{k_{gf}A_{gf}}{l_{gf}m_{Cu,1}c_{Cu,1}}\theta=-2\frac{k_{gf}A_{gf}}{l_{gf}\left(\rho Vc\right)_{Cu,1}}\theta$$

$$\int_{100}^{60}\frac{d\theta}{\theta}=-2\int_0^t\frac{k_{gf}A_{gf}}{l_{gf}\left(\rho Vc\right)_{Cu,1}}dt$$

At t = 0, $T_1 = 425$ K, $T_2 = 325$ K, $\theta(0) = 100$ K.
At t = t (the time we are trying to determine), $T_1 = 390$ K, which implies $T_2 = 360$ K and so $\theta(t) = 30$ K

Integrating, we get $\ln\left(\frac{30}{100}\right)=-2\frac{k_{gf}A_{gf}}{l_{gf}\left(\rho Vc\right)_{Cu,1}}t$. Solving, we get t = 110419 s = 1.278 days

Problem 4

a) The heat equation in spherical coordinates is:

$$\rho c\frac{\partial T}{\partial t}=k\nabla^2T+\dot{q}_v=k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\right)T+\dot{q}_v$$

By symmetry, the temperature does not depend on θ or ϕ. There is no time dependence, since the system is operating in a steady state. Therefore, the heat equation reduces to

$$0=k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\dot{q}_v$$

This can also be derived by applying the first law of thermodynamics to the infinitesimal shell (yellow colored) shown in the figure below. At steady state, the first law reduces to $0=\frac{dE}{dt}=\dot{Q}-\dot{W}=\dot{Q}$

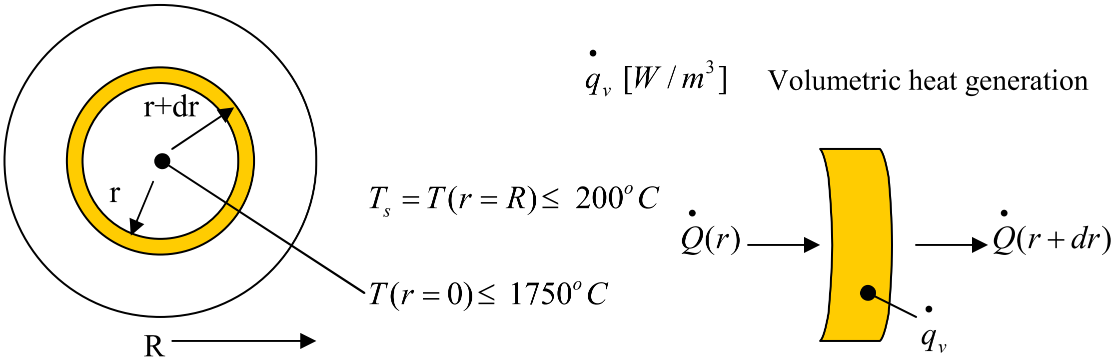
$$\therefore\ \dot{Q}(r) - \dot{Q}(r+dr) + \dot{q}_v \cdot \left(\text{Volume of the shell}\right) = 0$$

By Fourier's Law, $\dot{Q}(r) = \left(-k\dfrac{dT}{dr}\right)4\pi r^2$

$$\dot{Q}(r+dr) = \dot{Q}(r) + \frac{d}{dr}\left(\dot{Q}(r)\right)\cdot dr = \left(-k\frac{dT}{dr}\right)4\pi r^2 + \frac{d}{dr}\left(\left(-k\frac{dT}{dr}\right)4\pi r^2\right)\cdot dr$$

$$\therefore\ -\frac{d}{dr}\left(\left(-k\frac{dT}{dr}\right)4\pi r^2\right)\cdot dr + \dot{q}_v \cdot 4\pi r^2 dr = 0$$

$\therefore\ k\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{dT}{dr}\right) + \dot{q}_v = 0$ which is identical to the equation derived from the heat equation.



b) Solve the ODE.

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\dot{q}_v}{k}\cdot r^2$$

$$r^2\frac{dT}{dr} = -\frac{\dot{q}_v}{3k}\cdot r^3 + C_1 \ \Rightarrow\ \frac{dT}{dr} = -\frac{\dot{q}_v}{3k}\cdot r + \frac{C_1}{r^2}$$

$$\therefore T = -\frac{\dot{q}_v}{6k}\cdot r^2 - \frac{C_1}{r} + C_2$$

But the temperature should be finite as r approaches to zero. C_1 is thus zero.

$$\therefore T = -\frac{\dot{q}_v}{6k}\cdot r^2 + C_2$$

Let $T(r=R) = T_s$. Then $T_s = -\dfrac{\dot{q}_v}{6k}\cdot R^2 + C_2 \ \Rightarrow\ C_2 = T_s + \dfrac{\dot{q}_v}{6k}\cdot R^2$

$$\therefore T(r) = T_s + \frac{\dot{q}_v}{6k}(R^2 - r^2)$$

Note that maximum temperature is found at the origin (r = 0).

c) Maximum allowable volumetric heat generation

From $T(r=0) = T_s + \dfrac{\dot{q}_v}{6k}R^2$, we know $\dot{q}_v = \dfrac{6k}{R^2}\left(T(r=0) - T_s\right)$.

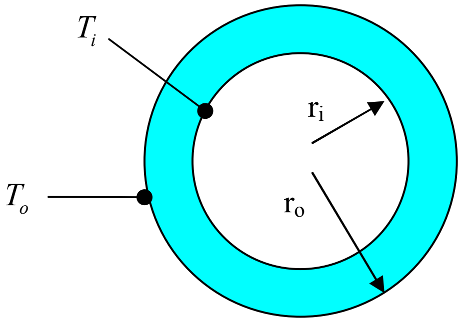
Keep in mind the fact that the heat transfer to the thermoelectric devices is the greatest when T_s is the highest. Hence we choose T_s = 200°C. The maximum temperature (which happens to be at the origin) cannot exceed 1750°C. Therefore the maximum allowable volumetric heating rate is such that the temperature at the origin is 1750°C.

$$\dot{q}_{v,\max} = \frac{6\times(4\ W/mK)}{(0.02\ m)^2}(1750^\circ C - 200^\circ C) = 93\ MW/m^3$$

d) efficiency = 4%

Power = $\eta\dot{q}_{v,\max}V_{total} = 0.04\left(93\ MW/m^3\right)\left(\frac{4}{3}\pi(0.02\ m)^3\right) = 124.66\ W$

e) Sheath



Boundary conditions

$T(r=r_i)=T_i$

$T(r=r_o)=T_o$

There is no heat generation in the sheath, and the temperature still does not change with time or angular coordinates. The governing equation does not differ from the result in part a) except that $\dot{q}_v$ =0

$$\therefore \frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$$

f) Plug in the new boundary conditions into the general solution of the ODE derived in part b) and set $\dot{q}_v$ =0.

From $T(r) = -\frac{\dot{q}_v}{6k}\cdot r^2 - \frac{C_1}{r} + C_2 = -\frac{C_1}{r} + C_2$,

$T_i = -\frac{C_1}{r_i} + C_2$ and $T_o = -\frac{C_1}{r_o} + C_2$.

$$\therefore C_1 = -\frac{r_o r_i\left(T_i - T_o\right)}{\left(r_o - r_i\right)},\ C_2 = \frac{r_o T_o - r_i T_i}{\left(r_o - r_i\right)} \Rightarrow T(r) = \frac{1}{r}\left(\frac{r_o r_i\left(T_i - T_o\right)}{\left(r_o - r_i\right)}\right) + \left(\frac{r_o T_o - r_i T_i}{\left(r_o - r_i\right)}\right)$$

g) The total heat transfer is calculated by applying Fourier's Law.

$$\dot{Q}\Big|_{r=r_i} = \left(-k\frac{dT}{dr}\right)_{r=r_i}\cdot 4\pi r_i^2 = \left[\frac{k}{r^2}\left(\frac{r_o r_i\left(T_i - T_o\right)}{\left(r_o - r_i\right)}\right)\right]_{r=r_i}\cdot 4\pi r_i^2 = 4\pi k\left(\frac{r_o r_i\left(T_i - T_o\right)}{\left(r_o - r_i\right)}\right)$$

(We get the same answer no matter what radius we evaluate $\dot{Q}$ at)

h) Thermal resistance

$$\dot{Q} = 4\pi k\left(\frac{r_o r_i\left(T_i - T_o\right)}{\left(r_o - r_i\right)}\right) = \frac{\left(T_i - T_o\right)}{\frac{1}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_o}\right)} \qquad \therefore R = \frac{1}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_o}\right)$$

We already know the thermal resistance for a cylindrical pipe. $R = \frac{\ln(r_o/r_i)}{2\pi L k}$, where L is the length of the pipe.