24.903 Language & Structure III: Semantics and Pragmatics
Spring 2003, 2-151, MW 1-2.30
February 24, 2003

1 The λ-notation for Functions

A traditional way of representing functions:
$sq(x) = x^2$

function-name, variable, definition

The **Lambda notation** arranges things differently:
$sq = \lambda x[x^2]$

$\lambda x[x^2]$ is called a λ-term. The structure of λ-terms is:
λ variable [. . . (variable) . . .]

- the variable inside the square brackets is bracketed because its presence is not obligatory.

- the value returned by the function is **whatever** the body of the λ-term, i.e. the expression within the square brackets evaluates to. This **whatever** could be a truth value (0, 1), a number, an individual, a set, a function etc.

λ **abstraction**: the process of creating a λ term from an expression potentially containing a variable. The variable could be over anything - a truth value, a number, an individual, a set, a function etc.

λ **conversion**: the λ term is a function. When this function is applied to an argument, the resulting value can be computed by replacing the occurences of the (outermost) λ variable in the expression in square brackets by the argument.

2 More on Conversion

E[M/x] means '**E** with **M** substituted for all instances of x'.

2.1 α-Conversion

used to rename variables in λ terms

$(\lambda x.[E])[y/x] \Longrightarrow (\lambda y.[E[y/x]])$

$(\lambda x.[x + 2])[y/x] \Longrightarrow$
$(\lambda y.[[x + 2][y/x]]) \Longrightarrow$
$(\lambda y.[y + 2])$

2.2 β-Conversion

used to apply functions, by replacing all instances of the variable in the body with the input.

$(\lambda x.[E])(y) \Longrightarrow E[y/x]$

$(\lambda x.[x^3])(2) \Longrightarrow$
$[x^3][2/x] \Longrightarrow 2^3$

2.3 η-Conversion

used to simplify functions

$(\lambda x.[f(x)]) \Longrightarrow f$

The term λ-conversion is often used as cover term for the above operations.

3 Functions with Restrictions

Function, defined in the traditional way, come with a domain specification.

Let $f_1 : Z \mapsto N$ be defined as $f_1(x) = x^2$, and
let $f_2 : N \mapsto N$ be defined as $f_2(x) = x^2$

Now f_1 and f_2 are not identical.
$f_1(-7) = 49$, $f_2(-7)$ is undefined.

Domain specification can be incorporated into λ terms straightforwardly. The following conventions are used:
$f_1 : \lambda x \in Z[x^2]$, $f_2 : \lambda x \in N[x^2]$
$f_1 : \lambda x[x \in Z|x^2]$, $f_2 : \lambda x[x \in N|x^2]$

More generally functions with restrictions using the λ-notation are represented as follows:
$\lambda\alpha : \phi.[\gamma]$

α is the argument variable,
ϕ the domain condition,
γ the value description.

4 Functions with Complex Arguments

Set arguments:
$\lambda X[X \cup \{a, b, c\}]$

Function arguments:
$\lambda f[f(2)]$
$\lambda f[f(2) + f(3)]$
$\lambda f[f(f(2) + f(3))]$

- Function application may not always work out -
$\lambda f[f(8 - f(4))](\lambda x[x \in N|x^2])$

The curious function: $\lambda f.[f(f)]$

5 Functions with more than one argument

$\lambda x.[\lambda y.[x^2 + y]]$
$\lambda x \lambda y.[x^2 + y]$

The role of variables:
$\lambda x[x^2] = \lambda y[y^2]$

Similarly
$\lambda x \lambda y[x^2 + y] = \lambda y \lambda x[y^2 + x]$, but
$\neq \lambda y \lambda x[x^2 + y]$
$\neq \lambda x \lambda x[x^2 + x]$

What does $\lambda x \lambda x[x^2 + x]$ mean?

What is the relationship between $\lambda x \lambda y[x^2 + y]$ and $\lambda y \lambda x[x^2 + y]$?

- The combinator C

6 The Scope of a Variable

In a lambda term $\lambda x[\ldots]$, the body of the λ-term $[\ldots]$ is the **scope** of x.

More precisely: the x after the λ **binds** any **free** instances of x in its scope.

free = not bound
Variable are born free - they get **bound** by the closest λ which has their name on it.

(1) a. $\lambda x[x^3 + x^2 + x + 1]$
All the x's in the square brackets are bound by the λx

b. $\lambda x.[x^3 + \lambda x.[x^2 + x + 1](x)]$
The x's inside the inner square brackets are bound by the inner λx. Only the argument to $\lambda x.[x^2 + x + 1]$ and x^3 are bound by the top level λx.

(1b) can be rewritten more clearly as (2).

(2) $\lambda x.[x^3 + \lambda y[y^2 + y + 1](x)]$
doing some λ conversion, we get
$\lambda x.[x^3 + x^2 + x + 1]$

Moral of the story: the names of variables are not important. What matters are the dependencies between argument positions (within the square brackets) and the order in which those arguments are supplied.

λ-conversion should not create spurious dependencies.

(3) Common Pitfalls:

a. Undoing a dependency:
$\lambda x.[x + (\lambda x.[x + 2])(y)](7) \not\Longrightarrow$
$[7 + (\lambda x.[7 + 2])(y)]$

b. Creating a spurious dependency:
$\lambda x.[x + (\lambda y.[x + y])(2)](y) \not\Longrightarrow$
$[x + (\lambda y.[y + y](2)]$

Auxiliary Moral: Variable names aren't important, but they can cause confusion. Hence when possible use α-conversion to eliminate re-use of variable names.

7 Function Composition

Let $f : A \to B$ and $g : B \to C$, then the result of composing f with g is written as $g \circ f$, $g \circ f : A \to C$, and $(g \circ f)(x) = g(f(x))$

$succ : N \to N$, $succ(x) = x + 1$
$sq : N \to N$, $sq(x) = x^2$
What is $succ \circ sq$, $sq \circ succ$?

Using the λ calculus, we can define a λ term that given functions f and g computes $f \circ g$:
$\lambda f[\lambda g[\lambda x[f(g(x))]]]$

• The combinator B

(4) 'maternal grandfather' as the composition of *mother* function with the *father* function
Swedish *morfar*, *farfar*, *farmor*, *mormor*

8 Characteristic Functions as λ terms

The characteristic function f_A corresponding to a set A can be defined as follows:

(5) $f_A : U \to \{0, 1\}$
$x \to 1$ if $x \in A$
$x \to 0$ if $x \notin A$

We can represent characteristic functions as λ terms:

(6) $f_A : \lambda x[x \in A]$

The characteristic function for the set $B = \{x : x \geq 17$ and x is even $\}$ can be written as:
$f_B = \lambda x[x \geq 17$ and x is even$]$

- the 'predicative' part of the set description and the body of a λ term are very similar.

- the one difference is that λ terms may have restrictors. For certain values, they may be undefined. This is unlike sets, where you're either in or out.