24.903 Language & Structure III: Semantics and Pragmatics
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1 Statement Logic

- sometimes also called the Propositional Calculus

- concerned with the logical behavior of statements, without looking beyond the clausal level.

1.1 Syntax

(1) We assume that there is a countably infinite number of basic statements, given by $p_1, p_2, p_3, \ldots$ In general, we will use p, q, r etc. to refer to arbitrary statements.

The only syntactic category is that of **well-formed formulas**, also called **wff**'s.

(2) Recursive definition of the set of all wff's of SL:

a. Basic Clause: every atomic statement is a wff.

b. Recursion clauses
  i. If Φ is a wff, then $\neg\Phi$ is a wff.
  ii. If Φ and Ψ are wff's, then so are:
  $[\Phi \wedge \Psi]$, (conjunction, Φ *and* Ψ; also written &)
  $[\Phi \vee \Psi]$, (disjunction, Φ *or* Ψ)
  $[\Phi \rightarrow \Psi]$, (conditional, *if* Φ *then* Ψ, also written $\supset$)
  $[\Phi \leftrightarrow \Psi]$, (biconditional, Φ *if and only if* Ψ, also written $\equiv$)

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ are called **logical constants, logical connectives, and logical variables**.

1.2 Semantic Rules

In SL, we assume that statements can be true (1) or false (0).

Then using truth tables, we can define the semantics of the logical connectives $\neg, \wedge, \vee, \rightarrow$, $\leftrightarrow$.

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Another **or**: **xor**, $\otimes$

p	q	$p \otimes q$
0	0	0
0	1	1
1	0	1
1	1	0

1.3 Computing Truth Conditions

• We can use truth tables to compute the truth conditions of any complex SL statement.

(3) Three kinds of statements
  a. Contingencies
  b. Tautologies
  c. Contradictions

1.4 Logical Consequence and Equivalence

(4) Logical Consequence
  a. Definition: ψ follows logically from ϕ iff **whenever** ϕ is true, then ψ is true.
  b. Notation: $\Rightarrow$.
  $\phi \Rightarrow \psi$ iff $\phi \rightarrow \psi$ is a tautology.

(5) Logical Equivalence
  a. Definition: ψ is logically equivalent to ϕ iff
  whenever ϕ is true, then ψ is true and
  whenever ψ is true, then ϕ is true.
  b. Notation: $\Leftrightarrow$.
  $\phi \Leftrightarrow \psi$ iff $\phi \leftrightarrow \psi$ is a tautology.

(6) Replacement:
  If two expressions ϕ and ψ are logically equivalent, and if ϕ occurs as a subformula in some complex formula, then we can replace an occurrence of ψ by ϕ, without changing the truth value of the complex formula.

2 Statement Logic Laws

T is a tautology and $\bot$ is a contradiction.

(7) a. Idempotency

  b. Associativity

  c. Commutativity

  d. Identity

  e. Complement:
  $[\phi \vee \neg\phi] \Leftrightarrow T$ (excluded middle)
  $\neg\neg\phi \Leftrightarrow \phi$ (double negation)
  $[\phi \wedge \neg\phi] \Leftrightarrow \bot$
  $\neg \bot \Leftrightarrow T$

  f. De Morgan

  g. Conditional Laws:
  $[\phi \rightarrow \psi] \Leftrightarrow [\neg\phi \vee \psi]$
  $[\phi \rightarrow \psi] \Leftrightarrow [\neg\psi \rightarrow \neg\phi]$ (Contraposition)

  h. Biconditional Laws:
  $[\phi \leftrightarrow \psi] \Leftrightarrow [[\phi \rightarrow \psi] \wedge [\psi \rightarrow \phi]]$
  $[\phi \leftrightarrow \psi] \Leftrightarrow [[\phi \wedge \psi] \vee [\neg\psi \wedge \neg\phi]]$

• Note the similarities between set theory and statement logic. They have the same underlying mathematical structure (a Boolean Algebra).

• a difference:
$A \subseteq B$ vs. $\phi \rightarrow \psi$