Discrete Character of Meson Masses

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Mesons of a given family show an approximately linear dependence of squared mass on angular momentum. Regge has expressed such trajectories in the mass squared – angular momentum plane as a multiple set of curves. These lines are analogous to the hydrogen atom energy level – angular momentum relationship. Further investigation shows that these sets of multiple lines can be reduced to a single line representing an entire meson family. In this work, the entire set of multiple lines representing the light meson family is replaced by a single line which arises naturally when the squared mass of each meson depends linearly on the squared mass of the pion. This relation is analogous to the Rydberg formula for the hydrogen spectrum, with the electron mass replaced by the pion mass.

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Regge plots have played an important role in the study of mesons for nearly half a century. In 1959, T. Regge [1-3] pioneered the study of poles in scattering theory. Chew and Frautschi [4], comparing the theory of Regge poles to observation, noticed that the squared mass of the then-known mesons varied linearly with the total angular momentum. Subsequently discovered mesons proved to lie either on this original line or on other lines parallel to it in mass squared - angular momentum space (with some possible slight deviations from linearity at large mass). In the 1960s and 1970s, reggeon exchange theory, in which all members of a given Regge trajectory are considered to represent a single quantum field, was often invoked in attempts to explain the strong interaction [4, 5]. The success of the constituent quark model reduced interest in this theoretical approach, but predicting the existence and shape of Regge trajectories remains a test of the success of any phenomenological meson model [6-9]. In particular, approximately-linear Regge trajectories have been shown to arise from one of the earliest string models of the meson [10], and their existence is still regarded as evidence for the view of the meson as a pair of quarks connected by a gluonic string.

Figure 1a shows the Regge trajectories of those mesons composed of u and d quarks. (Other families of mesons, with different constituent quarks, have qualitatively similar Regge diagrams but with different values for the slopes and intercepts of the lines.) The data points in published Regge diagrams tend to be scattered rather widely around the trajectories, with errors in some cases as large as 13%.

The large size of the error in linear Regge fits is well-known [11, 12], and is due to the complicated nature of the inter-quark potential. Most discussions of the meson Regge plot begin with an underlying theoretical model, and use reggeon exchange theory to derive ideal trajectories for comparison with experiment. The harmonic oscillator potential, for example, predicts evenly spaced, precisely linear trajectories [5], the straight lines shown in Figure 1a, which only very roughly fit the experimental data (represented by dots in Figure 1a). More sophisticated models [5] use a potential such as the Cornell potential

$$V = -\frac{4}{3}\alpha_s/r + \sigma r \tag{1}$$

where σ is the string tension and α_s is the strong fine-structure function (not actually a constant). These models also take into account the expected very large hyperfine splitting of meson mass levels. Calculated trajectories are in reasonable agreement with experiment, but again with large error bars.

It is of course also possible to regard the Regge plot as simply giving an empirical relationship between a particle's angular momentum and its rest mass or energy. This relationship need not be assumed *a priori* to have any particular form, such as the direct proportionality observed in mesons.

As an example, we consider the familiar case of the (idealised) non-relativistic hydrogen atom, ignoring fine and hyperfine structure. It is of course well-known that the energy levels of hydrogen to this approximation are given by the Rydberg formula:

$$E_n = -\frac{\alpha^2 m_e c^2/2}{n^2}, \tag{2}$$

where α is the fine structure constant, c the speed of light, and m_e the reduced mass of the electron. On the other hand, the orbital angular momentum quantum number l, which does not appear in the energy formula, varies from 0 to $n-1$. Thus a plot of n against l, or equivalently of $1/\sqrt{|E_n|}$ against l, would produce a collection of straight-line trajectories, as shown in Figure 2a. Each trajectory of the figure has the same slope $\alpha(m_e c^2/2)^{1/2}$; the intercepts, which correspond to atomic s orbitals, are regularly spaced along the y-axis.

Note that all the points on each line have the same value of $n-l-1$. Since the wave function in the Coulomb problem has n nodes, of which $l+1$ are nodes of the angular part, it follows that each trajectory of Figure 2a links states having the same number of nodes in the purely radial part of the wave function.

Figure 2a makes no direct reference to n; nevertheless, it is obvious by inspection of Figure 2a that n exists, because the data points are located only at integer values of the y coordinate. Ths observation would be enough to reconstruct the Rydberg formula from Figure 2a alone. The linear relationship between $1/\sqrt{|E_n|}$ and n is shown in Figure 2b.

With real experimental data, even if every spectroscopic level were to be observed, the situation would be more complicated because of the lifted l-degeneracy. If the system under study were, for example, positronium, with fine structure splittings nearly as large as the splittings between n levels, the equivalent of Figure 2a might prove bafflingly complex. However, for hydrogen, such splittings are small, so that the existence of n and its approximately linear relationship to $1/\sqrt{|E_n|}$ could still be easily deduced.

We suggest that meson Regge trajectories be treated in exactly the same way as hydrogen in the example just given. For mesons, the role of $1/\sqrt{|E_n|}$ will be played by m^2, and that of the orbital angular momentum quantum number l by the total angular momentum quantum number j. One would not expect in advance that there be any equivalent of the principal quantum number n in this analogy, since there is no accepted formula for meson masses analogous to the Rydberg formula for the energy levels of hydrogen.

Nevertheless, a close inspection of Regge diagrams suggests that the data points tend to line up in horizontal rows, very much like the horizontal rows of points in Figure 2a. Such an alignment implies that the meson mass spectrum is composed of almost uniformly spaced levels in m^2.

The quantisation of the squared mass of the light mesons is illustrated in Figure 1b, which is the counterpart of Figure 2b. The squared masses of the 44 mesons containing only u, d, or s quarks are clearly seen from the figure to be uniformly spaced at intervals of about 0.3 $(\mathrm{GeV})^2$, so that

$$m_q^2 = 0.3q + 0.084 \tag{3}$$

where q is an integer and m_q the mass of any light meson. Eq. (3) is the light meson analogue of the hydrogen Rydberg formula, Eq. (2).

Most theoretical approaches to the meson spectrum [13-17] do not predict integer quantisation of the squared mass. An exception is the semi-relativistic model of Sergeenko [18]. In his model, a universal meson mass formula resembling our Eq. (3) is obtained by interpolating between non-relativistic and ultra-relativistic limiting cases; the resulting formula predicts a linear dependence of squared mass on the principal quantum number n. The same prediction is also made in theoretical work currently in progress, using both the one-body Dirac equation and a five-dimensional model [19].

Unfortunately, the integer q in Eq. (3) is not proportional to n as predicted by Sergeenko and others. The observed meson masses are spread around the predicted values with as much as 70% error.

To interpret Eq. (3), we note that the slope of the best-fit line is found to be 0.3 $(\mathrm{GeV})^2$, and the intercept is 0.084 $(\mathrm{GeV})^2$. On the other hand, the squared mass of the pion is 0.018225 $(\mathrm{GeV})^2$ [11]. Thus the slope of the best-fit line is about 16 times the squared mass of the pion, and the intercept is about 4 times the squared mass of the pion. Assuming that this is not a coincidence, we suggest replacing Eq. 3 with

$$m_q = 2m_\pi\sqrt{4q-1} = 0.27\sqrt{4q-1} \tag{4}$$

where m_π is the mass of the pion. Eq. 4 can be squared to give

$$m_q^2 = 0.29q + 0.073 \tag{5}$$

which is almost the same as the empirical best-fit given by Eq. (3). Notwithstanding the 13% difference in the intercept, the meson masses predicted by Eq. (5) are almost as close to the experimental data as those obtained by evaluating the empirical formula Eq. (3).

Table 1 shows that the meson masses predicted by Eq. (4) still agree quite well with the experimental data. In all 44 of the cases investigated, the observed meson masses [11] have values lying within 10% of those predicted by Eq. (4), that is, well within the error range seen in Regge plots. All of the mesons except the $\rho(770)$ have masses within 5% of the predicted values. Only 15 observed masses are off by more than 0.01 GeV, and only 16 differ by 0.02 GeV or less from the predicted values.

Returning briefly to the analogy with the hydrogen atom, we note that, upon inserting the numerical value of the fine-structure constant, the Rydberg formula, Eq. (2), becomes (in units such that $c = 1$):

$$E_n = -(5.328 \times 10^{-5}) m_e \sqrt{4((2n)^{-4})} \tag{6}$$

which resembles Eq. (4) insofar as the energy levels are proportional to basic mass, in this case that of the electron. Of course, the proportionality in the case of the hydrogen atom has its origin in the electromagnetic interaction, with its fine structure constant 7.3×10^{-3}. In the case of the meson, the proportionality coefficient is naturally much larger, having its origin in the strong nuclear force [20].

In summary, we have shown that if the masses of mesons composed of u, d, and s quarks are proportional to the mass of the pion (in analogy with the hydrogen atomic levels which are proportional to the electron mass), then the proportionality constant for each meson squares to an odd integer $4q - 1$. This is the meson equivalent of the Rydberg formula for hydrogen.

In other words, we are saying that the hydrogen energy levels originate in the electromagnetic interaction of the constituents of the hydrogen atom, namely an electron and a proton, in a systematic manner proportional to the electron mass, as predicted by the Rydberg formula, Eq. (2). By the same token, the masses of the various mesons originate in the strong interaction of the constituent quarks of the meson, in a systematic manner proportional to the pion mass, as predicted by the Rydberg-like formula given by Eq. (4).

Table 1: Light Meson Masses from Eq. 4

q	Resonance	Observed Mass (GeV)	Predicted Mass (GeV)	Percent Error
1	$K^{\pm}$	0.49	0.47	+4.26
2	ρ	0.77	0.71	+8.45
3	K	0.89	0.89	0.00
4	ϕ	1.02	1.05	-2.86
5	a_1	1.23	1.18	+4.24
5	b_1	1.23	1.18	+4.24
6	K_1	1.27	1.29	-1.55
6	π	1.30	1.29	+0.78
6	a_2	1.32	1.29	+2.33
7	f_0	1.37	1.40	-2.14
7	K_1	1.40	1.40	0.00
7	K_0	1.41	1.40	+0.71
7	K	1.41	1.40	+0.71
7	$K_2^{\pm}$	1.43	1.40	+2.14
7	a_0	1.45	1.40	+3.57
8	ρ	1.46	1.50	-2.67
8	K	1.46	1.50	-2.67
8	f_1	1.51	1.50	+0.67
8	f_2	1.52	1.50	+1.33
10	π_2	1.67	1.69	-1.18
10	ϕ	1.68	1.69	-0.59
10	ρ_3	1.69	1.69	0.00
10	ρ	1.70	1.69	+0.59
10	K	1.72	1.69	+1.78
11	K_2	1.77	1.77	0.00
11	K_3	1.78	1.77	+0.56
11	π	1.80	1.77	+1.69
12	K_2	1.82	1.85	-1.62
12	K	1.83	1.85	-1.08
12	ϕ_3	1.85	1.85	0.00
14	K_2	1.98	2.00	-1.00
14	f_2	2.01	2.00	+0.50
15	ρ_4	2.04	2.07	-1.45
15	K_4	2.05	2.07	-0.97
15	π_2	2.09	2.07	+0.97
16	ρ	2.15	2.14	+0.47
17	K_2	2.24	2.21	+1.36
18	ρ_3	2.25	2.27	-0.88
19	K_3	2.32	2.34	-0.85
19	ρ_5	2.33	2.34	-0.43
19	f_2	2.34	2.34	0.00
20	K_5	2.38	2.40	-0.83
21	ρ_6	2.45	2.46	-0.41
21	K_4	2.49	2.46	+1.22

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Figure 1: Light meson squared masses as function of (a) total angular momentum quantum number J, and (b) integer q.

Figure 2: Hydrogen atom energy levels as function of (a) angular momentum quantum number l, and (b) principal quantum number n.