Away-step Frank-Wolfe Method for Convex Optimization Involving a Log-Homogeneous Barrier

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Motivating Example: D-Optimal Design

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where the stepsize $\alpha_k \geq 0$ is given by exact line-search.

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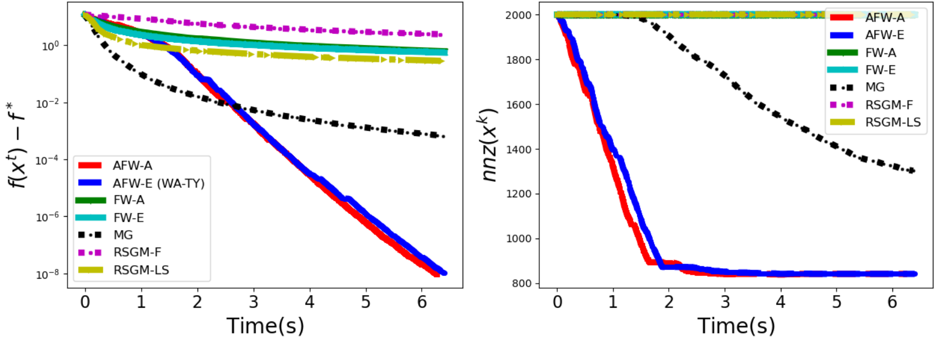

ASFW-A & ASFW-E (this work): Away-step FW methods for LHB
FW-A & FW-E [Fed72; Kha96; ZFce]: Generalized FW methods for LHB
RSGM-F & RSGM-LS [BBT17; LFN18]: Relatively smooth gradient method
MG [STT78]: Multiplicative gradient method

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- Some deeper questions:
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- In this work, we will provide affirmative answers to the questions above.

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- Besides D-optimal design, other applications include
  - Budget-constrained D-optimal design
  - Positron emission tomography
  - (Reformulated) Poisson image deblurring with TV-regularization

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  1. $\left|D^3 f(y)[w,w,w]\right| \leq 2\|w\|_y^3 \quad \forall\, y \in \mathsf{int}\,\mathcal{K}, \forall\, w \in \mathbb{Y}$,
  2. $f(y_k) \to +\infty$ for any $\{y_k\}_{k\geq 1} \subseteq \mathsf{int}\,\mathcal{K}$ such that $y_k \to u \in \mathsf{bd}\,\mathcal{K}$,
  3. $f(ty) = f(y) - \theta\ln(t) \quad \forall\, y \in \mathsf{int}\,\mathcal{K}, \forall\, t > 0$.

  where $\|w\|_y := \langle \nabla^2 f(y)w, w\rangle^{1/2}$ denotes the local norm of w at $y \in \mathsf{int}\,\mathcal{K}$.

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  - (Choose direction) If $|\mathcal{S}_k| = 1$ or $G_k > \widetilde{G}_k$, let $d^k := d_{\mathrm{F}}^k$ and $\bar{\alpha}_k := 1$; otherwise, let $d^k := d_{\mathrm{A}}^k$ and $\bar{\alpha}_k := \beta_{a^k}^k/(1-\beta_{a^k}^k)$.

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  - (Choose stepsize) Choose $\alpha_k \in (0, \bar{\alpha}_k]$ in one of the following two ways:
    - Adaptive stepsize: Compute $r_k := -\langle \nabla F(x^k), d^k\rangle$ and $D_k := \|\mathsf{A}d^k\|_{y^k}$. If $D_k = 0$, then $\alpha_k := \bar{\alpha}_k$; otherwise, $\alpha_k := \min\{b_k, \bar{\alpha}_k\}$, where $b_k := r_k/(D_k(r_k + D_k))$.
    - Exact line-search: $\alpha_k \in \arg\min_{\alpha_k\in(0,\bar{\alpha}_k]} F(x^k + \alpha d^k)$.

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  - (Update iterates) Update $x^{k+1} := x^k + \alpha_k d^k$ and $\beta^{k+1} \in \Delta_{|\mathcal{V}|}$ such that $x^{k+1} = \sum_{v\in\mathcal{V}} \beta_v^{k+1} v$, and let $\mathcal{S}_{k+1} := \mathrm{supp}(\beta^{k+1})$.

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- For all applications of interest, computing $D_k = \|\mathsf{A} d^k\|_{y^k} = \langle \nabla^2 F(x^k) d^k, d^k \rangle^{1/2}$ takes $O(n)$ times, instead of $O(n^2)$ time.

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$$\delta_k \leq (1-\rho)^{k_{\text{eff}}}\delta_0, \quad \text{where} \quad \rho := \min\left\{\frac{1}{5.3(\delta_0+\theta+B)}, \frac{\mu\Phi(\mathcal{X},\mathcal{X}^*)^2}{42.4(\theta+B)^2}\right\},$$

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- All the quantities defining ρ are *affine-invariant* and *norm-independent*.

Computational Guarantees

Global linear convergence of $\{G_k\}_{k\geq 0}$:

For some (affine-invariant) $\bar{D} < +\infty$ and all $k \geq 0$, we have

$$G_k \leq \begin{cases} 4(1-\rho)^{k_{\text{eff}}}\delta_0 \max\{\bar{D}, 1\}, & \text{if } \delta_k > 1 \\ 4\sqrt{1-\rho}^{k_{\text{eff}}}\sqrt{\delta_0}\max\{\bar{D}, 1\}, & \text{if } \delta_k \leq 1 \end{cases}.$$

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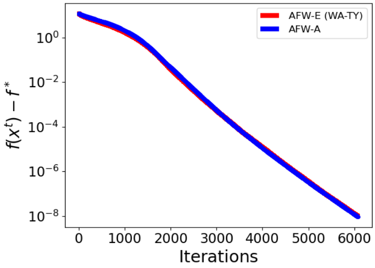

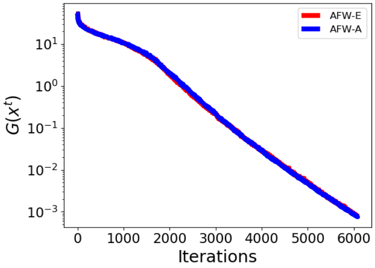

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For all $k \geq \bar{k}$, if $x^k \notin \mathcal{F}$, then

- $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$, when either exact line-search or adaptive stepsize is used in Step 7,
- $\mathcal{S}_{k+1} = \mathcal{S}_k \setminus \{a^k\}$ for some $a^k \in \mathcal{S}_k \cap \bar{\mathcal{V}}_{\mathcal{F}}$, when exact line-search is used in Step 7;

otherwise, if $x^k \in \mathcal{F}$, then $x^l \in \mathcal{F}$ for all $l \geq k$.

Another Example: Positron Emission Tomography

$$\max_{x\in\Delta_n}\ \left\{F(x) := \sum_{j=1}^m p_j \ln(a_j^\top x)\right\} \qquad \texttt{(PET)}$$

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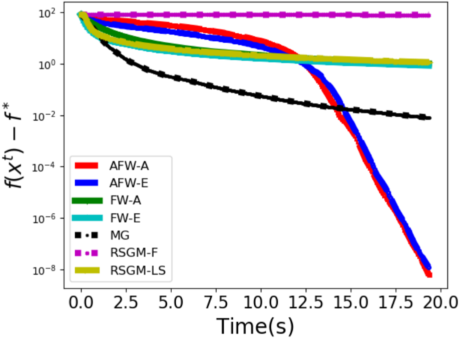

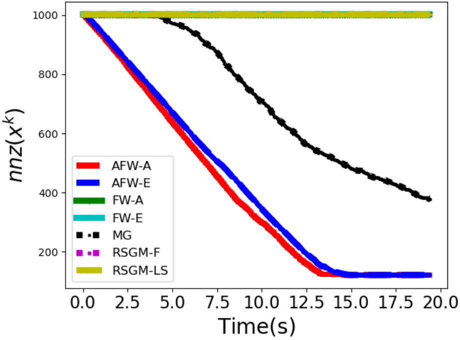

Thank you!

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