SYMMETRY POINTS OF CONVEX SETS: Basic Properties and Computational Complexity

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Measuring Symmetry

Let $S \subset \mathbb{R}^n$ be convex body, that is, a compact convex set with nonempty interior.

$$\text{sym}(\bar{x}, S) := \max \{ t \mid y \in S \Rightarrow \bar{x} - t(y - \bar{x}) \in S \}$$
Symmetry Points, $x^*$

$$\text{sym}(S) := \max_{\bar{x} \in S} \text{sym}(\bar{x}, S)$$

$S$ is symmetric if $\text{sym}(S) = 1$

$x^*$ is called a symmetry point of $S$ if $\text{sym}(x^*, S) = \text{sym}(S)$
Motivation for studying $\text{sym}(S)$

- symmetry properties of $S$ are invariant under affine transformation of $S$

- $\text{sym}(x, S)$ measures distance from $x$ to $\partial S$ independent of any particular norm $\| \cdot \|$

- $\text{sym}(x, S)$ is continuous w.r.t. $x$ and $S$

($\text{sym}(S) \geq \frac{1}{n}$, follows from the Löwner-John theorem)
Motivation for studying \( \text{sym}(S) \), continued

- \( \text{sym}(\cdot, S) \) appears in the theoretical complexity of interior-point methods for optimization over \( S \):

\[
O \left( \sqrt{\vartheta} \ln \left( \left( \frac{1}{\text{sym}(x^0, S)} \right) + \vartheta + \cdots \right) \right)
\]

- We intend to use symmetry concepts to create theoretical/practical pre-conditioners for interior-point methods for linear and conic optimization.
Motivation for studying $\operatorname{sym}(S)$, continued

- Relation to classical convex geometry
- Proximity to the centroid (which is hard to compute)
- Extension of results that assume $\operatorname{sym}(S) = 1$
Set Inclusion Formulation

\[ \text{sym}(\bar{x}, S) = \max \{ t \mid y \in S \Rightarrow \bar{x} - t(y - \bar{x}) \in S \} \]

\[ = \max_{\alpha} \alpha \]

s.t. \[ \alpha(\bar{x} - S) \subset (S - \bar{x}) \]
Set Inclusion Formulation, continued

\[
\text{sym}(S) = \max_{\alpha, x} \alpha \\
\text{s.t.} \quad \alpha(x - S) \subset (S - x)
\]
Outline

• Simple Set Operations and Symmetry

• Some Geometric Properties

• Characterization of Symmetry Points

• Computing $\text{sym}(S')$
Simple Set Operations
Simple Set Operations and Symmetry

Let $x \in S$ and $y \in T$, then

- Superminimality under Intersection:
  \[ \text{sym}(x, S \cap T) \geq \min \{ \text{sym}(x, S), \text{sym}(x, T) \} \]

- Superminimality under Addition:
  \[ \text{sym}(x + y, S + T) \geq \min \{ \text{sym}(x, S), \text{sym}(x, T) \} \]
Simple Set Operations and Symmetry, continued

- Invariance under Polarity:

  \[ \text{sym}(0, S - x) = \text{sym}(0, (S - x)^\circ) \]

- Minimality under Cartesian Product:

  \[ \text{sym}((x, y), S \times T) = \min \{ \text{sym}(x, S), \text{sym}(y, T) \} \]

- Lower bound under affine transformations.
  Let \( A(\cdot) \) be an affine transformation. Then

  \[ \text{sym}(A(x), A(S)) \geq \text{sym}(x, S) \]
Some Geometric Properties
Some Geometric Properties

- $\text{sym}(x, S)$ is a quasiconcave function of $x$
  
  \[
  \{x \in \mathbb{R}^n \vert \text{sym}(x, S) \geq t\} \text{ is a convex set}
  \]

- If $S$ is symmetric, $\text{sym}(x, S)$ is logconcave
Some Geometric Properties, continued

Let \( x, y \in S \), \( \alpha = \text{sym}(x, S) \) and \( \beta = \text{sym}(y, S) \)

- Bound on distance,

\[
\|x - y\| \leq \left( \frac{1 - \alpha \beta}{1 + \alpha + \beta + \alpha \beta} \right) \text{diam}(S)
\]

- Bound on cross-ratio distance,

\[
d_S(x, y) \leq \frac{1}{\alpha \beta} - 1
\]
Fix an arbitrary $v \in \mathbb{R}^n$. Define

\[ H^+(x) := \{z \in S : \langle z, v \rangle \leq \langle x, v \rangle \} \]

\[
\frac{\text{sym}(x, S)^n}{1 + \text{sym}(x, S)^n} \leq \frac{\text{Vol}(H^+(x))}{\text{Vol}(S)} \leq \frac{1}{1 + \text{sym}(x, S)^n}
\]
Some Geometric Properties, continued

\[ H(x) := \{ z \in S : \langle z, v \rangle = \langle x, v \rangle \} \]

Consider the \((n - 1)\)-dimensional volume of \(H(x)\).
From the Brunn-Minkowski inequality,

\[ f(x) = \text{Vol}_{n-1} \left( H(x) \right)^{\frac{1}{n-1}} , \]

is a concave function.
Theorem: Let \( x^* \) be a symmetry point of \( S \), then

\[
\frac{f(x^*)}{\max_{y \in S} f(y)} \geq \frac{2 \text{sym}(S)}{1 + \text{sym}(S)}
\]
Some Geometric Properties, continued

\( P \) is a \( \beta \)-approximation of \( S \) if there exists \( x \) for which

\[
\beta P \subset S - x \subset P
\]

- If \( P \) is a \( \beta \)-approximation \( S \), and \( \text{sym}(P) = \alpha \), then

\[
\text{sym}(S) \geq \beta \alpha
\]

- Let \( B(x, r) \) denote the ball centered at \( x \) with radius \( r \). Then,

\[
B(x, r) \subset S \subset P \subset S + B(0, \delta) \Rightarrow \text{sym}(x, S) \geq \text{sym}(x, P) \left( 1 - \frac{\delta}{r} \right)
\]
Some Geometric Properties, continued

• For any $x \in S$, there exists an ellipsoid $E$ centered at $x$ such that

$$E \subset S \subset \frac{\sqrt{n}}{\text{sym}(x, S)}E$$

• Furthermore, if $x^L \in S$ be the center of a Löwner-John pair, then

$$E \subset S \subset \sqrt{\frac{n}{\text{sym}(x, S)}}E$$
Extension of Löwner-John Theorem: There exists an ellipsoid $E$ that is a $\beta$-approximation of $S$ for

$$\beta \leq \min \left\{ \sqrt{\frac{n}{\text{sym}(x^L, S)}}, \sqrt{\frac{n}{\text{sym}(S)}}, n \right\}.$$  

Conjecture: There exists an ellipsoid $E$ that is a $\beta$-approximation of $S$ for

$$\beta \leq \sqrt{\frac{n}{\text{sym}(S)}}.$$
Integration over a Convex Set

\( X \in \mathbb{R}^n, \quad X \sim N(0, \Sigma) \)

\( Y \in \mathbb{R}^n \) be an arbitrary random variable

\( S \subset \mathbb{R}^n \) be a compact convex set, \( 0 \in \text{int} \ S \)

**Theorem (Anderson 1955)**

Suppose \( \text{sym}(0, S) = 1 \). Then, for \( \beta \in [0, 1] \),

\[
P(X + \beta Y \in S) \geq P(X + Y \in S)
\]

**Extension:** Let \( \text{sym}(0, S) = \alpha > 0 \). Then

\[
P(X + \beta Y \in S) \geq \alpha^n P \left( X + \frac{Y}{\alpha} \in S \right)
\]
Characterization of Symmetry Points
Characterization of Symmetry Points

Recall

\[
\text{sym}(\bar{x}, S) = \max_\alpha \alpha \\
\text{s.t.} \quad \alpha(\bar{x} - S) \subset (S - \bar{x})
\]
Characterization of Symmetry Points via Touching Points and Normal Cone

Let $\alpha = \text{sym}(x, S)$, we define

**Touching points $V(x)$:**

\[
V(x) := (x - \alpha(S - x)) \cap \partial S
\]

$(V(x) \neq \emptyset$ by definition of $\alpha)$
Characterization of Symmetry Points via Touching Points and Normal Cone, continued

Denote the normal cone to $S$ at $x$ as

$$N_S(x) = \{ s \in \mathbb{R}^n : \langle y - x, s \rangle \leq 0, \text{for all } y \in S \}$$

Support Vectors of touching points

$$SV(x) = \{ s \in \mathbb{R}^n : (v, s) \in V(x) \times N_S(v), \|s\| = 1 \}$$
Recall $x^*$ is a symmetry point of $S$ if $\text{sym}(x^*, S) = \text{sym}(S)$.

The following are equivalent:

- $x^*$ is a symmetry point of $S$
- $0 \in \text{conv} SV(x^*)$

If $S$ is a strictly convex set, then $x^*$ is unique.
Define $Opt(S) = \{ x \in S : \text{sym}(x, S) = \text{sym}(S) \}$

- Klee (1953) proved that

$$\frac{1}{\text{sym}(S)} + \text{dim}(Opt(S)) \leq n$$
Computing $\text{sym}(S)$
Computing $\text{sym}(S)$

Recall

$$\text{sym}(\bar{x}, S) = \max_\alpha \alpha$$

s.t. $\alpha(\bar{x} - S) \subset (S - \bar{x})$

In general, evaluating $S_1 \subset S_2$ is a hard problem, even when $S_1, S_2$ are convex
Computing $\text{sym}(S)$ for polyhedral $S$

\[ S = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \text{ where } A \text{ is } m \times n \]
Lemma Suppose that every solution \( x \) of \( Mx \leq g \) satisfies \( Px \leq q \). Then there exists a matrix \( \Pi \) satisfying:

\[
\begin{align*}
\Pi M &= P \\
\Pi g &\leq q \\
\Pi &\geq 0
\end{align*}
\]

(“\( \Pi \geq 0 \)” is component-wise on the \( m \times m \) matrix \( \Pi \) )
Computing \( \text{sym}(S) \) via One LP

\[
(SP) \quad \text{sym}(S) = \max_{\Pi, \alpha, x} \alpha \\
\text{s.t.} \quad \Pi A = -\alpha A \\
\Pi (b - Ax) \leq \alpha (b - Ax) \\
\Pi \geq 0
\]

\[
(LP1) \quad \frac{1}{\text{sym}(S)} = \min_{\Lambda, \gamma, w} \gamma \\
\text{s.t.} \quad -\Lambda A = A \\
\Lambda b + Aw \leq b\gamma \\
\Lambda \geq 0
\]
Computing $\text{sym}(S)$ via One LP

Recover $\text{sym}(S) = \frac{1}{\gamma^*}$ and $x^* = \frac{1}{1+\gamma^*w^*}$

$LP1$ has $m^2 + m$ linear inequalities and $mn + m$ equations

**Drawback:** Complexity of each Newton Step $O(m^6)$!
Computing $\text{sym}(S)$ via $m + 1$ LPs

For $i = 1, \ldots, m$, solve:

$$(LP_i) \quad \delta_i^* := \max_x b_i - A_i x$$

s.t. \hspace{1cm} Ax \leq b

Then solve:

$$(LP2) \quad \frac{1}{\text{sym}(S)} = -1 + \min_{x, \theta} \theta$$

s.t. \hspace{1cm} Ax - b\theta \leq -\delta^* \hspace{1cm} \theta \geq 0

Let $(\hat{x}, \hat{\theta})$ solve $(LP2)$. Then $x^* \leftarrow \frac{\hat{x}}{\theta}$ is a symmetry point of $S$
Complexity of $(m + 1)$-LPs Method

This scheme involves solving $m + 1$ LPs, each of size $m \times n$

What is the theoretical complexity of approximately computing $\text{sym}(S)$ using these LPs?
Recall the *analytic center* problem for $S$:

\[
\text{ACP : } \quad \min_x \ f(x) := -\sum_{i=1}^{m} \ln(b_i - A_i x)
\]

\[\text{s.t. } \quad Ax < b\]

An *approximate* analytic center $\hat{x}$ satisfies:

\[
\sqrt{\nabla f(\hat{x})^T [\nabla^2 f(\hat{x})]^{-1} \nabla f(\hat{x})} \leq \frac{1}{2}
\]
Complexity of \((m + 1)\)-LPs Method, continued

**Definition:** \(\bar{x}\) is an \(\epsilon\)-symmetry point of \(S\) if

\[
sym(\bar{x}, S) \geq (1 - \epsilon)\text{sym}(S)
\]

Suppose we know an approximate analytic center \(\hat{x}\) of \(Ax \leq b\). Then:

**Theorem** An \(\epsilon\)-symmetry point of \(S\) can be computed in at most

\[
7m^{1.5} \ln \left( \frac{13m}{\epsilon} \right)
\]

iterations of Newton’s method.
Projective Transformations

A projective transformation of a set $S$ is given by

$$y = T(x) = \frac{Mx + g}{p^T x + \zeta}$$

where $p^T x + \zeta > 0$ for all $x \in S$
Projective Transformations

How can projective transformations be used to transform $S$ to $T(S)$ in such a way that $\text{sym}(T(S)) > \text{sym}(S)$?

Can we find a projective transformation $T(\cdot)$ that optimizes $\text{sym}(T(S))$?