# Summary of the area calculations

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Summary of the area calculations.

PACS numbers:

The area of a black hole is given by

$$A = 8\pi M^2 \left(1 + \sqrt{1 - \chi^2}\right) \tag{0.1}$$

where $M$ is the black hole mass and $\chi = S/M^2$.

We are interested in the change of area between the initial system of two black holes and the final state of one black hole:

$$\delta A = A_f - A_i \tag{0.2}$$

The error in a quantity can be derived from known errors in other quantities by using the propagation of errors formula. If $x = x(u, v, \ldots)$, then

$$\sigma_x^2 \approx \sigma_u^2 \left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2 \left(\frac{\partial x}{\partial v}\right)^2 + \cdots + 2\sigma_{uv}^2 \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \cdots \tag{0.3}$$

where $\sigma_u^2$ is the variance of $u$, $\sigma_v^2$ is the variance of $v$, and $\sigma_{uv}^2$ is the covariance of $u$ and $v$. In our usual notation, these are the elements of the covariance matrix $\Sigma_{uu}$, $\Sigma_{vv}$, and $\Sigma_{uv}$.

For $\delta A$, we can first see that

$$(\Delta(\delta A))^2 = (\Delta A_i)^2 + (\Delta A_f)^2 \tag{0.4}$$

where the covariance term is zero because the measurements are taken independently during the inspiral and ringdown.

The initial area is

$$A_i = 8\pi m_1^2 \left(1 + \sqrt{1 - \chi_1^2}\right) + 8\pi m_2^2 \left(1 + \sqrt{1 - \chi_2^2}\right) \tag{0.5}$$

so we need these derivatives:

$$\frac{\partial A_i}{\partial m_1} = 16\pi m_1 \left(1 + \sqrt{1 - \chi_1^2}\right) \tag{0.6}$$

$$\frac{\partial A_i}{\partial m_2} = 16\pi m_2 \left(1 + \sqrt{1 - \chi_2^2}\right) \tag{0.7}$$

$$\frac{\partial A_i}{\partial \chi_1} = \frac{-8\pi m_1^2 \chi_1}{\sqrt{1 - \chi_1^2}} \tag{0.8}$$

$$\frac{\partial A_i}{\partial \chi_2} = \frac{-8\pi m_2^2 \chi_2}{\sqrt{1 - \chi_2^2}} \tag{0.9}$$

giving

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$$
\begin{aligned}
(\Delta A_i)^2 = \quad & 256\pi^2 m_1^4 \left(1+\sqrt{1-\chi_1^2}\right)^2 \left(\frac{\Delta m_1}{m_1}\right)^2 + 256\pi^2 m_2^4 \left(1+\sqrt{1-\chi_2^2}\right)^2 \left(\frac{\Delta m_2}{m_2}\right)^2 + \frac{64\pi^2 m_1^4 \chi_1^2}{1-\chi_1^2}(\Delta\chi_1)^2 \\
+ \; & \frac{64\pi^2 m_2^4 \chi_2^2}{1-\chi_2^2}(\Delta\chi_2)^2 - 256\pi^2 m_1^4 \chi_1 \frac{1+\sqrt{1-\chi_1^2}}{\sqrt{1-\chi_1^2}} \Sigma_{\ln m_1,\chi_1} - 256\pi^2 m_2^4 \chi_2 \frac{1+\sqrt{1-\chi_2^2}}{\sqrt{1-\chi_2^2}} \Sigma_{\ln m_2,\chi_2} \\
- \; & 256\pi^2 m_1^2 m_2^2 \chi_2 \frac{1+\sqrt{1-\chi_1^2}}{\sqrt{1-\chi_2^2}} \Sigma_{\ln m_1,\chi_2} - 256\pi^2 m_1^2 m_2^2 \chi_1 \frac{1+\sqrt{1-\chi_2^2}}{\sqrt{1-\chi_1^2}} \Sigma_{\ln m_2,\chi_1} \\
+ \; & 512\pi^2 m_1^2 m_2^2 \left(1+\sqrt{1-\chi_1^2}\right)\left(1+\sqrt{1-\chi_2^2}\right) \Sigma_{\ln m_1,\ln m_2} + \frac{128\pi^2 m_1^2 m_2^2 \chi_1 \chi_2}{\sqrt{1-\chi_1^2}\sqrt{1-\chi_2^2}} \Sigma_{\chi_1,\chi_2}
\end{aligned}
\tag{0.10}
$$

We also have

$$
A_f = 8\pi M_f^2 \left(1+\sqrt{1-\chi_f^2}\right) \tag{0.11}
$$

from which we can get

$$
\frac{\partial A_f}{\partial M_f} = 16\pi M_f \left(1+\sqrt{1-\chi_f^2}\right) \tag{0.12}
$$

$$
\frac{\partial A_f}{\partial \chi_f} = \frac{-8\pi M_f^2 \chi_f}{\sqrt{1-\chi_f^2}} \tag{0.13}
$$

so that

$$
\begin{aligned}
(\Delta A_f)^2 = \quad & 256\pi^2 M_f^4 \left(1+\sqrt{1-\chi_f^2}\right)^2 \left(\frac{\Delta M_f}{M_f}\right)^2 + \frac{64\pi^2 M_f^4 \chi_f^2}{1-\chi_f^2}(\Delta\chi_f)^2 \\
- \; & 256\pi^2 M_f^4 \chi_f \frac{1+\sqrt{1-\chi_f^2}}{\sqrt{1-\chi_f^2}} \Sigma_{\ln M_f,\chi_f}
\end{aligned}
\tag{0.14}
$$

Here we make the approximations

$$
\frac{\Delta M_f}{M_f} \approx \frac{2}{\rho} \tag{0.15}
$$

$$
\Delta\chi_f \approx \frac{2.5}{\rho} \tag{0.16}
$$

$$
\Sigma_{\ln M_f,\chi_f} = c_{\ln M_f,\chi_f} \left(\frac{\Delta M_f}{M_f}\right)(\Delta\chi_f) \approx 0.972\frac{5}{\rho^2} \tag{0.17}
$$

where $\rho$ is the ringdown SNR and the approximations are made using Finn, figures 2 and 3a. These are the complete equations used in the code to calculate $\Delta(\delta A)$. Of course, there are some conversions to be done. All covariance matrix elements have to be divided by the inspiral SNR squared because of the conventions of the code. Also, all these error terms are multiplied by the factor

$$
R_\odot^2 = \frac{G^2}{c^4} \tag{0.18}
$$

to convert areas from square solar masses to square kilometers.