MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.901: Astrophysics I — Spring Term 2005

PROBLEM SET 10 SOLUTIONS

1. Star formation: the collapse of a molecular cloud

$$\rho\ddot{r} = -\frac{GM_r\rho}{r^2} - \frac{dP}{dr}$$

$T = 150\,\mathrm{K}$, $n = 10^8\,\mathrm{cm}^{-3}$, $\rho = n\mu m_H$

$$M_J = \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} = 17.0\,M_\odot$$

$$R_J = \left(\frac{15kT}{4\pi G\mu m_H\rho_0}\right)^{1/2} = 2.43\times10^3\,\mathrm{AU}$$

a) $$\left|\frac{dP}{dr}\right| \approx \left|\frac{\Delta P}{\Delta r}\right| \sim P_c/R_J = \frac{\rho kT}{\mu m_H}\cdot\left(\frac{4\pi G\mu m_H\rho}{15kT}\right)^{1/2}$$

$$= nkT\left(\frac{4\pi G}{15}\right)^{1/2}(\mu m_H)\left(\frac{n}{kT}\right)^{1/2}$$

$$= \left(\frac{4\pi G}{15}\right)^{1/2}(\mu m_H)\,n^{3/2}(kT)^{1/2}$$

$$\boxed{\left|\frac{dP}{dr}\right| \approx 6\times10^{-23}\ \mathrm{dyne\cdot cm^{-3}}}$$

b) Assuming hydrostatic equilibrium,

$$\frac{dP}{dr} = \frac{-GM_r\rho}{r^2} \approx \frac{-GM_J\cdot M_J}{\frac{4}{3}\pi R_J^5} = \frac{-GM_J^2}{\frac{4}{3}\pi R_J^5}$$

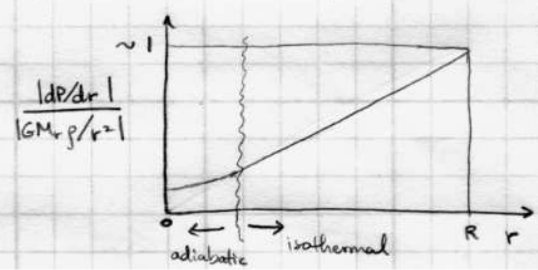
$$\boxed{\left|\frac{dP}{dr}\right| \approx 3\times10^{-22}\ \mathrm{dyne\cdot cm^{-3}}}$$

c) As the gas collapses, we can estimate

$$\frac{|dP/dr|}{|GM_r\rho/r^2|} \approx \frac{P_c/r}{GM_J\,\rho/r^2} = \frac{\cancel{\rho}kT/\cancel{r}\mu m_H}{GM_J\cancel{\rho}/r^{\cancel{2}}} = \underbrace{\frac{kT}{GM_J\mu m_H}}_{\substack{\text{constant}\\\text{during}\\\text{collapse}\\\text{if isothermal}}}\cdot r$$

$$\boxed{\frac{|dP/dr|}{|GM_r\rho/r^2|} \propto r}$$

Since the radius will shrink by several orders of magnitude during collapse, the pressure gradient can be neglected soon after the start of the collapse.



2. Magnetic fields and star formation

Using the cloud from problem 1, $M = 17.0\,M_\odot$, $R = 2.43\times10^3$ AU

$$\mathcal{U} = -\frac{3}{5}\frac{GM^2}{R} = -1\times10^{45}\ \text{erg}$$

$$\boxed{u_{grav} = \mathcal{U}/\tfrac{4}{3}\pi R^3 = -6\times10^{-6}\ \text{erg}\cdot\text{cm}^{-3}}$$

Assuming $B = 10\,\mu G$ throughout the cloud,

$$\boxed{u_{EM} = \frac{B^2}{8\pi} = 4\times10^{-12}\ \text{erg}\cdot\text{cm}^{-3}}$$

Apparently $|u_{EM}/u_{grav}| \ll 1$, so magnetic field effects are not likely to be important.

3. Angular momentum and star formation

a) Including the centrifugal term, the equation of motion becomes

$$\rho\ddot{r} = \frac{-GM_r\rho}{r^2} + \frac{\rho v_\phi^2}{r} - \frac{dP}{dr},$$

where v_ϕ is the azimuthal velocity. The angular momentum per unit mass is $\ell \equiv r v_\phi$, so

$$\rho\ddot{r} = \frac{-GM_r\rho}{r^2} + \rho\frac{\ell^2}{r^3} - \frac{dP}{dr}$$

Using $\frac{d^2r}{dt^2} = v_r\frac{dv_r}{dr}$, this can be integrated to yield a statement of energy conservation, which can also just be written down:

$$E = \frac{-GM_r\rho}{r} + \frac{1}{2}\rho v_r^2 + \frac{1}{2}\rho\frac{\ell^2}{r^2} = \text{const}$$

Equating the initial and final energies:

$$\cancelto{0}{\frac{-GM_r\rho}{r_0}} + \frac{1}{2}\rho\cancelto{0}{v_{r,0}^2} + \cancelto{0}{\frac{1}{2}\rho\frac{\ell^2}{r_0^2}}$$

$$= \frac{-GM_r\rho}{r_f} + \frac{1}{2}\rho\cancelto{0}{v_{r,f}^2} + \frac{1}{2}\rho\frac{\ell^2}{r_f^2}$$

$$\Rightarrow \frac{GM_r}{r_f} = \frac{1}{2}\frac{\ell^2}{r_f^2} = \frac{1}{2}\frac{\omega_0^2 r_0^4}{r_f^2}$$

$$\Rightarrow \boxed{r_f = \frac{\omega_0^2 r_0^4}{2GM_r}}$$

b) Assume $M = 1M_\odot$, $r_0 = 0.5\,\text{pc}$, $r_f = 100\,\text{AU}$, then

$$\boxed{\omega_0 = \left(\frac{2GM\,r_f}{r_0^4}\right)^{1/2} = 2.6\times10^{-16}\ \text{rad}\cdot\text{s}^{-1}}$$

$$\boxed{v_\phi = \omega_0 r_0 = 4.1\times10^2\ \text{cm}\cdot\text{s}^{-1}}$$

c) When collapse begins, assume a uniform sphere; when done, assume uniform disk.

$$I_{\text{sphere}} = \frac{2}{5}MR^2$$
$$I_{\text{disk}} = \frac{1}{2}MR^2$$

By conservation of angular momentum:

$$\frac{2}{5}\cancel{M}r_0^2\omega_0 = \frac{1}{2}\cancel{M}r_f^2\omega_f \Rightarrow \boxed{\omega_f = \frac{4}{5}\left(\frac{r_0}{r_f}\right)^2\omega_0 = 2.2\times10^{-10}\ \text{rad}\cdot\text{s}^{-1}}$$

$$\boxed{v_\phi = \omega_f r_f = 3.3\ \text{km}\cdot\text{s}^{-1}}$$

d) $\boxed{P_{orb} = \frac{2\pi}{\omega_f} = 9.1 \times 10^2 \text{ yr}}$

From Keplers 3rd law, ($P^2 = a^3/M$ in solar-centric units), expect

$$\boxed{P_{Kep} = \left(\frac{a^3}{M}\right)^{1/2} = \left(\frac{100^3}{1}\right)^{1/2} = 1.0 \times 10^3 \text{ yr}}$$

The two periods are not identical because the protostar is rotating as a rigid body (in our simplified treatment). They are very similar, however, since the collapse will halt when the outer layers reach the Keplerian orbital frequency and hence the centrifugal barrier.

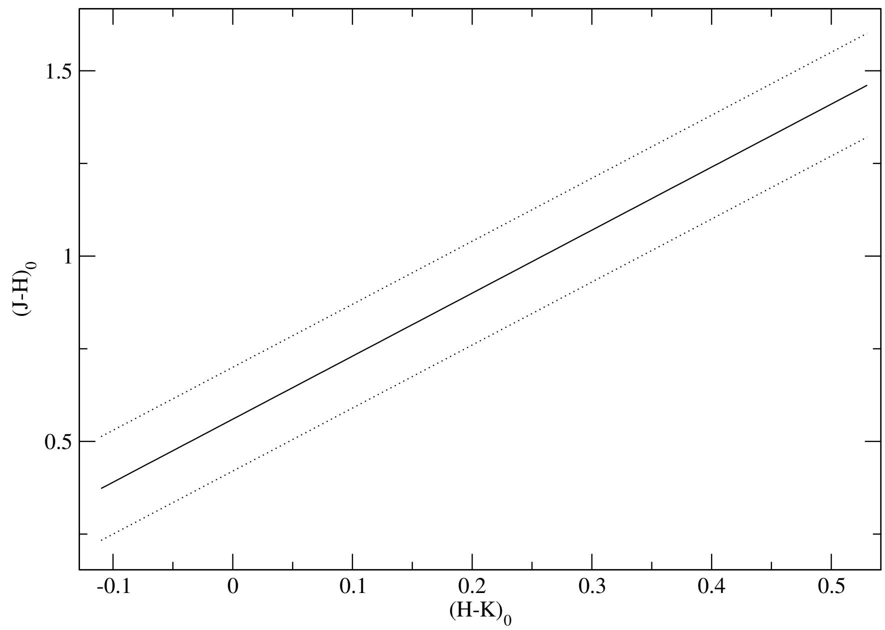


Figure 1: $(J-H)_0$ vs. $(H-K)_0$ with error bars, for the star observed in problem 4.

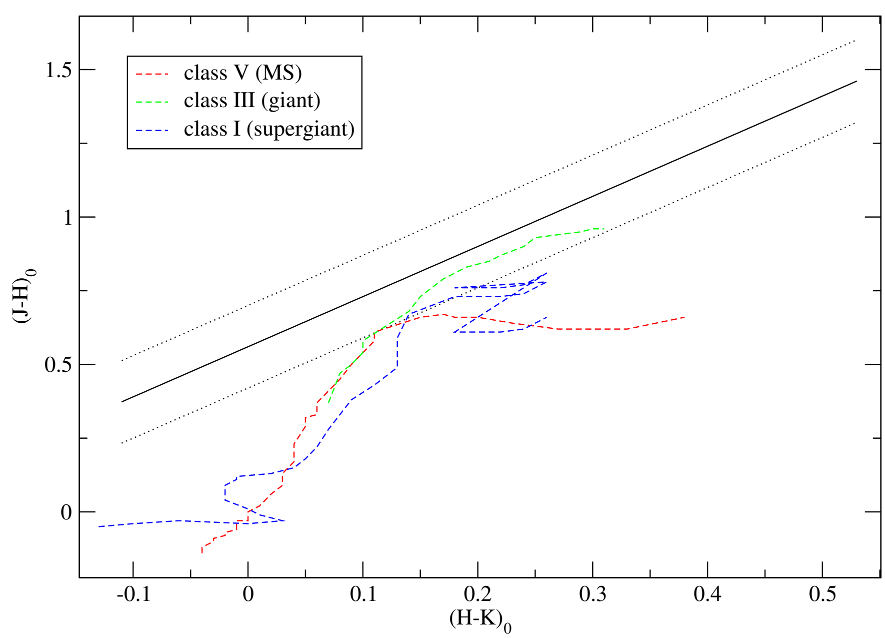


Figure 2: $(J-H)_0$ vs. $(H-K)_0$, with the intrinsic colors of luminosity class I, III, and V stars superimposed.

4. Deducing the nature of a highly reddened star

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Infrared observations of a star lying toward the Galactic center, with

$J = 19.21 \pm 0.05$ $A_J/A_V = 0.282$ $J-H = 3.05 \pm 0.07$

$H = 16.16 \pm 0.05$ $A_H/A_V = 0.175$ $H-K = 1.47 \pm 0.07$

$K = 14.69 \pm 0.05$ $\underbrace{A_K/A_V = 0.112}_{\text{reddening}}$

a) Recall

$$m_{\lambda,0} = m_\lambda - A_\lambda$$

So that

$$(J-H)_0 = J-H-A_J+A_H = 3.05\pm0.07 - 0.107\,A_V = \begin{cases} 1.45\pm0.07 & A_V=15 \\ 0.38\pm0.07 & A_V=25 \end{cases}$$

$$(H-K)_0 = H-K-A_H+A_K = 1.47\pm0.07-0.063\,A_V = \begin{cases} 0.53\pm0.07 & A_V=15 \\ -0.11\pm0.07 & A_V=25 \end{cases}$$

$$(J-H)_0 = 3.05\pm0.07 - 0.107\,A_V$$

$$23.3\pm1.1 - 15.9\,(H-K)_0 = A_V$$

$$(J-H)_0 = 3.05\pm0.07 - 2.49\pm0.12 + 1.70\,(H-K)_0$$

$$\boxed{(J-H)_0 = 1.70(H-K)_0 + 0.56\pm0.14}$$

b) From the figure, our star is most likely a class III (giant), although it could also plausibly be a class V (MS) or class I (supergiant).

Class	H-K range	A_V range	Spectral range
V	0.11-0.13	21.2-21.5	K4-K5
III	0.10-0.31	18.4-21.7	K0-M7
I	0.14-0.21	20.0-21.1	K5-M1 Iab
			M0-M1 Ib
			M0-M1 Ia

c) the distance of the star is given by

$$m_\lambda = M_\lambda - 2.5\log_{10}\left(f(d)/f(10pc)\right) = M_\lambda + 5\log_{10}\left(d/10pc\right)$$

$$d = 10pc\; 10^{(m_\lambda - M_\lambda)/5}$$

where m_λ is the de-reddened apparent magnitude and M_λ is the absolute.

Class	Spectral type	M_K	A_V	A_K	$K-A_K$	d (kpc)
V	K4	4.40	21.5	2.41	12.28	0.38
	K5	4.50	21.5	2.41	12.28	0.36
III	K0	-1.61	21.7	2.43	12.26	5.9
	K1	-1.90	21.7	2.43	12.26	6.8
	K2	-2.20	21.4	2.40	12.29	7.9
	K3	-2.73	21.1	2.36	12.33	10.3
	K4	-3.23	20.9	2.34	12.35	13.1
	K5	-3.80	20.6	2.31	12.38	17.2
	M0	-4.25	20.3	2.27	12.42	21.6
	M1	-4.55	20.0	2.24	12.45	25.1
	M2	-4.90	19.8	2.22	12.47	29.8
	M3	-5.14	19.5	2.18	12.51	33.9
	M4	-5.50	19.3	2.16	12.53	40.4
	M5	-6.26	18.7	2.09	12.60	59.2
	M6	-7.14	18.5	2.07	12.62	89.5
	M7	-8.10	18.4	2.06	12.63	140
I	K5 Iab	-9.30	21.1	2.36	12.33	212
	M0 Iab	-9.40	20.4	2.28	12.41	230
	M1 Iab	-9.50	20.1	2.25	12.44	244
	M0 Ib	-9.40	20.4	2.28	12.41	230
	M1 Ib	-9.50	20.1	2.25	12.44	244
	M0 Ia	-9.40	20.4	2.28	12.41	230
	M1 Ia	-9.50	20.1	2.25	12.44	244

most probable range

With the amount of extinction at $A_V \sim 20$, we expect our star to be somewhere near the Galactic center, which is ~8.5 kpc away. We can immediately discount the class V stars, which would only be $\lesssim 1$ kpc away. Similarly, we can discount anything more than ~10 kpc away, because it would be beyond the Galactic center, and thus more extincted than $A_V \sim 20$. Thus we conclude that the star is a K0-K3 giant.