

# LECTURES ON REAL OPTIONS: PART III — SOME APPLICATIONS AND EXTENSIONS

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- Now turn to some applications and extensions of the methodology developed so far.
  - Valuing an undeveloped offshore oil reserve, and determining when it should be developed.
  - Operating options: The option to shut down a factory when production is unprofitable, and the option to resume production later.
  - Sequential investment: Investment decisions made sequentially and in a particular order.
  - Jump processes: Key variable (e.g., price, value of a project) makes discrete jumps up or down. For example, a copper mine might be expropriated, or a machine might break down.

# Valuing Undeveloped Oil Reserves

- An undeveloped oil reserve is like a call option. It gives the owner the right to acquire a developed reserve by paying the development cost.
- When to develop the reserve? Same as deciding when to exercise a call option.
- The higher the uncertainty over future oil prices, the more valuable is the undeveloped reserve, and the longer we should wait to develop it.
- Use of option theory focuses attention on nature and extent of uncertainty.

# Comparison of Stock Call Option and Undeveloped Petroleum Reserve

## Stock Call Option

Current stock price

Variance of rate of return on the stock

Exercise price

Time to expiration

Riskless rate of interest

Dividend

## Undeveloped Reserve

Current value of developed reserve

Variance of rate of change of the value of a developed resource

Development Cost

Relinquishment requirement

Riskless rate of interest

Net production revenue less depletion

# Valuing Undeveloped Oil Reserves

- Undeveloped oil reserve is an option to "buy" a developed oil reserve, so first step is to determine value of developed reserve.
  - Value will depend on price of oil.
  - Price of oil fluctuates stochastically, so value of developed reserve will also fluctuate.
- Once we know the stochastic process for the value of developed reserve, use option theory to value the undeveloped reserve.

# Valuing Undeveloped Oil Reserves (continued)

## 1. Value of Developed Reserve

$B$  = number of barrels of oil

$V$  = value per barrel

$R$  = rate of return to owner of reserve

$P$  = price of a barrel of oil

$\Pi$  = After-tax profit from producing and selling a barrel of oil

$\mu$  = risk-adjusted expected rate of return on oil

$\omega$  = fraction produced each year = .10

- Production cost  $\approx .3P$ , tax rate  $\approx .34$ ,

$$\text{so } \Pi = (1 - .34)(.7)P = .45P$$

- $V = \int_0^{\infty} \omega \Pi e^{-(r+\omega)t} dt = \frac{\omega(.45P)}{r + \omega} = \frac{.045P}{.14} = .32P$

# Valuing Undeveloped Oil Reserves (continued)

- Depletion of reserve:  $dB = -\omega Bdt$
- Return dynamics:  $Rdt = \omega B\Pi dt + d(BV)$   
 $= \omega B\Pi dt + BdV - \omega BVdt$
- Return is partly random:  $Rdt/BV = \mu dt + \sigma dz$ ,  
where  $dz = \epsilon\sqrt{dt}$
- $\mu BVdt + \sigma BVdz = \omega B\Pi dt + BdV - \omega BVdt$ ,  
so  $dV = \mu Vdt - \omega(\Pi - V)dt + \sigma Vdz$
- Therefore:  $dV = (\mu - \delta)Vdt + \sigma Vdz$ , where  $\delta$  is payout rate net of depletion:

$$\delta = \omega(\Pi - V)/V \approx .04$$

## 2. Value of Undeveloped Reserve

- Given process for  $V$ , value of undeveloped reserve, we can now determine value of developed reserve.
- Let  $F(V, t)$  = value of 1-barrel unit of undeveloped reserve.
- By setting up risk-free portfolio, etc., can show that  $F$  must satisfy:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} + (r - \delta)V \frac{\partial F}{\partial V} - rF = -\frac{\partial F}{\partial t}$$



# Valuing Undeveloped Oil Reserves (continued)

- Boundary conditions:

$$F(0, t) = 0$$

$$F(V, T) = \max[V_T - D, 0]$$

$$F(V^*, t) = V^* - D$$

$$\partial F(V^*, t) / \partial V = 1$$

where  $D$  = per barrel cost of development

$V^*$  = critical value that triggers development

$T$  = time to expiration

# Valuation Example

- Developed reserve expected to yield 100 million barrels of oil;
- The present value of the development cost is \$11.79 per barrel;
- The development lag is 3 years;
- Relinquishment is after a period of 10 years;
- Standard deviation of value of developed reserves is 14.2 percent;
- The payout ratio (net production revenues/value of reserves) is 4.1 percent; and
- The value of the developed reserve today is \$12 per barrel.

Step 1: Calculate the present value of a developed reserve ( $V'$ ):

$$V' = e^{-(.041) \times 3} \times \$12 = \$10.61 \text{ per barrel.}$$

Step 2: Calculate the ratio of reserve value to development cost,  $C$ :

$$C = V'/D = \$10.61/\$11.79 = .90.$$

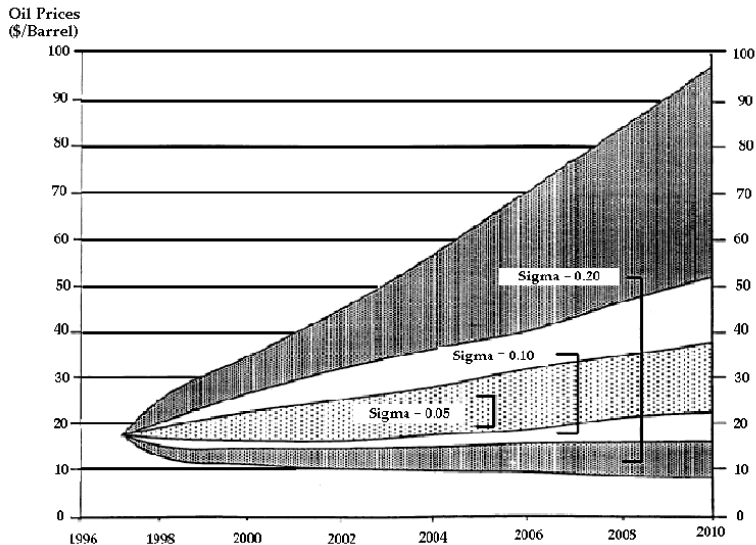
# Valuation Example (continued)

Step 3: Calculate the value of the undeveloped reserve:

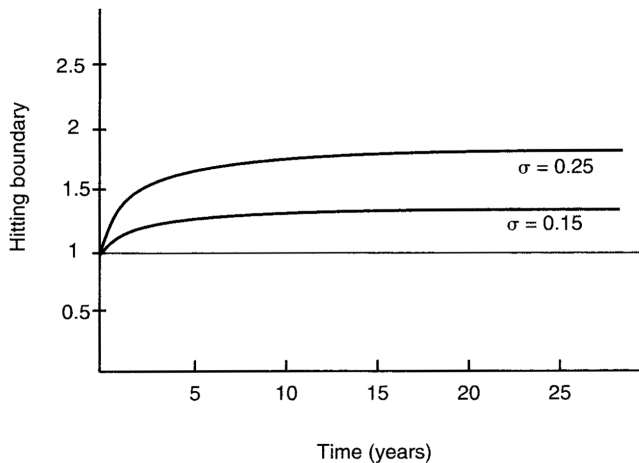
Value = (Option Value per \$1 Development Cost (from Table))  $\times$   
(Total Development Cost) =  $(0.05245) \times (\$1179.0 \text{ million}) =$   
\$61.84 million.

- Hence, although reserve cannot be profitably developed under current conditions, the right to develop it in the future is worth more than \$60 million.
- If you believe standard deviation of value of developed reserve is 25% instead of 14%, value of undeveloped reserve is much higher.
- So determining standard deviation is critical!

# Oil Price Uncertainty (90% Probability Range)



# Optimal Timing of Exercise



**Figure:** *Critical Value for Development of Oil Reserve*  
(Shows  $V^* / D$  for  $\delta = 0.04$  and  $r = 0.0125$ , where  $D$  is development cost)

# Option Values per \$1 of Development Cost

V/D	$\sigma_v = 0.142$			$\sigma_v = 0.25$	
	T=5	T=10	T=15	T=5	T=10
0.80	0.01810	0.02812	0.03309	0.07394	0.010392
0.85	0.02761	0.03894	0.04430	0.09174	0.12305
0.90	0.04024	0.05245	0.05803	0.11169	0.14390
0.95	0.05643	0.06899	0.07458	0.13380	0.16646
1.00	0.07661	0.08890	0.09431	0.15804	0.19071
1.05	0.10116	0.11253	0.11754	0.18438	0.21664
1.10	0.13042	0.14025	0.14464	0.21278	0.24424
1.15	0.16472	0.17242	0.17599	0.24321	0.27349

Source: Siegel, Smith, and Paddock (1987).

Note: Because option values are homogeneous in the development cost, total option value is the entry in the table times the total development cost.

# Undeveloped Reserve Values (in \$ Millions)

V/D	$\sigma = 14.2\%$			$\sigma = 25\%$	
	T=5	T=10	T=15	T=5	T=10
0.70	7.72	15.59	20.09	52.83	83.46
0.80	21.34	33.15	39.01	87.18	122.52
0.90	47.44	61.84	68.42	131.68	169.66
1.00	90.32	104.81	111.19	186.33	224.85
1.10	153.77	165.35	170.53	250.87	287.96

Note: This table uses a payout ratio of 4.1 percent and 100 Million Bbls of Oil.

# Is GBM the Right Model of Price?

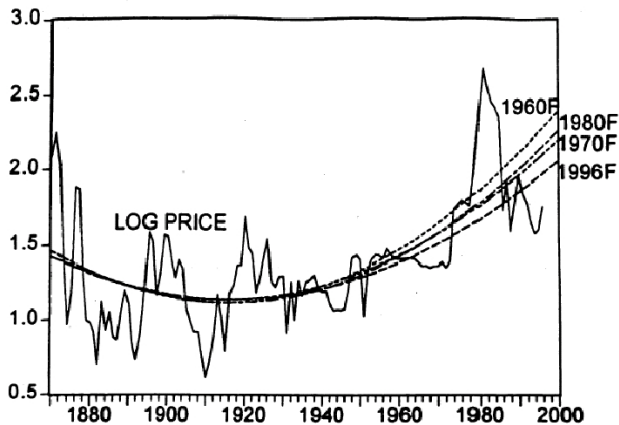


Figure: Log Price of Crude Oil and Quadratic Trend Lines



# Mean Reversion in Oil Price

- Consider the following mean-reverting process for the value of a developed oil reserve:

$$dV = \eta (\bar{V} - V) V dt + \sigma V dz. \quad (1)$$

- Then the partial differential equation for value of undeveloped reserve,  $F(V, t)$ , is

$$\frac{1}{2} \sigma_v^2 V^2 F_{VV} + [r - \mu + \eta(\bar{V} - V)] V F_V - r F = -F_t \quad (2)$$

- If the time until relinquishment is long enough (more than five years), we can ignore the time dependence of  $F(V, t)$ , so that the term  $-F_t$  disappears.
  - As we saw, solution can be written using the confluent hypergeometric function, which has a series representation.
  - Can use this solution, with different values for  $\eta$  and  $\bar{V}$ , to determine the extent to which mean reversion matters.

# Mean Reversion in Oil Price (continued)

- Wey (1993) has shown, using a 100-year series for the real price of crude oil, that a reasonable estimate for  $\eta$  is about 0.3, and that using this value (and a value of  $\sigma_v$  of .20), the extent to which mean reversion matters depends on the value to which  $V$  reverts,  $\bar{V}$ , relative to the development cost,  $D$ .
  - If  $\bar{V}$  is much larger than  $D$ , accounting for mean reversion gives a larger value for the undeveloped reserve when  $V < D$  because  $V$  is expected to rise over time. Wey shows that if  $\bar{V}$  is about twice as large as  $D$ , ignoring mean reversion can lead one to undervalue the reserve by 40 percent or more.
  - On the other hand, if  $\bar{V}$  is about as large as  $D$ , ignoring mean reversion will matter very little.

# Operating Options

- You own a factory that produces widgets: 1000/month for the next 10 years.
- Variable cost of production is  $C = \$10$  per widget.
- The price of widgets is now  $P = \$15$ , but  $P$  will fluctuate over time.
- What is the value of this factory?
- In each month, you have an option to produce 1000 widgets and receive  $\$1000P$ . For each option, the exercise price is  $1000C = \$10,000$ .
- You have  $10 \times 12 = 120$  European call options, one for each month.
- Let  $F_n(P)$  denote the value of the option to produce 1000 widgets in month  $n$  when the current price of widgets is  $P$ .

# Operating Options (continued)

- Then the value of this factory is simply the sum of the values of the 120 options, i.e., it is:

$$V = F_1(P) + F_2(P) + \dots + F_{120}(P)$$

- The Black-Scholes formula (modified for dividends) can be used to value each  $F_n(P)$ . To do this, one must determine the drift and volatility for  $P$ .
- Assume

$$dP/P = \alpha dt + \sigma dz$$

Let  $\mu$  = risk-adjusted expected return on  $P$ .

- Then  $\delta = \mu - \alpha$  is the “return shortfall.” It is equivalent to a dividend rate in the modified Black-Scholes formula. To find  $F_n(P)$ , just reduce  $P$  by the present value of the “dividends”, and apply the standard B-S formula.

## Operating Options (continued)

- For example, suppose  $P = \$15$ ,  $\alpha = 0$ ,  $\mu = r$ . Then,  $\delta = r$ , to find, say,  $F_{12}(P)$ , i.e., the value of the option to produce one year from now, replace  $P$  with

$$P' = P - rP/(1+r) = P/(1+r)$$

and then use the standard B-S formula:

$$F_{12} = P'N(d_1) - Ce^{-rT}N(d_2)$$

$$\text{where } d_1 = \frac{\ln(P'/C) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

- Here  $N(\cdot)$  is the cumulative probability distribution function for a standardized normal distribution. Also,  $T = 1$  if we are looking at the option to produce one year from now.

# Sequential Investment

- Many investment decisions are made sequentially, and in a particular order.
  - Oil production capacity: Find reserves, then develop them.
  - New line of aircraft: Engineering, prototype production, testing, final tooling.
  - Drug development: Find new molecule, Phase I testing, Phase II, Phase III, construct production facility, marketing.
  - Any investment that can be halted midway and temporarily or permanently abandoned.
- Like a compound option; each stage completed (or dollar invested) gives the firm an option to complete the next stage (or invest the next dollar).

# Sequential Investment (continued)

- Example: two-stage investment in new oil production capacity.
  - First, obtain reserves, through exploration or purchase, at cost  $I_1$ .
  - Second, Build development wells, at cost  $I_2$ .
  - Begin with option, worth  $F_1(P)$ , to invest in reserves. Investing gives the firm another option, worth  $F_2(P)$ , to invest in development wells.
  - Making this second investment yields production capacity, worth  $V(P)$ .
- Work backwards to find the optimal investment rules.

# Investment Rule for a Two-Stage Project

- Project, once completed, produces one unit of output per period at an operating cost  $C$ .
  - Output can be sold at price  $P$ , which follows a GBM:

$$dP = \alpha P dt + \sigma P dz. \quad (3)$$

- Production can be temporarily suspended when  $P$  falls below  $C$ , and resumed when  $P$  rises above  $C$ , so profit flow is  $\pi(P) = \max[P - C, 0]$ .
- Investing in first stage requires sunk cost  $I_1$ , and second stage requires sunk cost  $I_2$ .



# Investment Rule for a Two-Stage Project (con't.)

- Solve the investment problem by working backwards:
  - First, find value of completed project  $V(P)$ .
  - Next, find value of option to invest in second stage,  $F_2(P)$ , and critical price  $P_2^*$  for investing.
  - Then, find value of option to invest in first stage,  $F_1(P)$ , and critical price  $P_1^*$ .

# Investment Rule for a Two-Stage Project (con't.)

- Value of the Project.  $V(P)$  must satisfy

$$\frac{1}{2} \sigma^2 P^2 V''(P) + (r - \delta) P V'(P) - r V(P) + \pi(P) = 0, \quad (4)$$

subject to  $V(0) = 0$ , and continuity of  $V(P)$  and  $V_P(P)$  at  $P = C$ . Solution is:

$$V(P) = \begin{cases} A_1 P^{\beta_1} & \text{if } P < C, \\ B_2 P^{\beta_2} + P/\delta - C/r & \text{if } P > C. \end{cases} \quad (5)$$

where

$$\beta_1 = \frac{1}{2} - (r - \delta)/\sigma^2 + \sqrt{[(r - \delta)/\sigma^2 - \frac{1}{2}]^2 + 2r/\sigma^2} > 1,$$

$$\beta_2 = \frac{1}{2} - (r - \delta)/\sigma^2 - \sqrt{[(r - \delta)/\sigma^2 - \frac{1}{2}]^2 + 2r/\sigma^2} < 0.$$

# Investment Rule for a Two-Stage Project (con't.)

Constants  $A_1$  and  $B_2$  found from continuity of  $V(P)$  and  $V'(P)$  at  $P = C$ :

$$A_1 = \frac{C^{1-\beta_1}}{\beta_1 - \beta_2} \left( \frac{\beta_2}{r} - \frac{\beta_2 - 1}{\delta} \right) \quad (6)$$

$$B_2 = \frac{C^{1-\beta_2}}{\beta_1 - \beta_2} \left( \frac{\beta_1}{r} - \frac{\beta_1 - 1}{\delta} \right) \quad (7)$$

# Investment Rule for a Two-Stage Project (con't.)

- Second-Stage Investment. Find value of option to invest in second stage,  $F_2(P)$ , and critical price  $P_2^*$ .
  - Value of option must satisfy

$$\frac{1}{2} \sigma^2 P^2 F_2''(P) + (r - \delta) P F_2'(P) - r F_2(P) = 0 \quad (8)$$

subject to the boundary conditions

$$F_2(0) = 0 \quad (9)$$

$$F_2(P_2^*) = V(P_2^*) - I_2 \quad (10)$$

$$F_2'(P_2^*) = V'(P_2^*) \quad (11)$$

# Investment Rule for a Two-Stage Project (con't.)

- Guess and then confirm that  $P_2^* > C$ , so use solution for  $V(P)$  in eq. (5) for  $P > C$ , in conditions (10) and (11). From condition (9):

$$F_2(P) = D_2 P^{\beta_1}. \quad (12)$$

From boundary conditions (10) and (11),

$$D_2 = \frac{\beta_2 B_2}{\beta_1} (P_2^*)^{(\beta_2 - \beta_1)} + \frac{1}{\delta \beta_1} (P_2^*)^{(1 - \beta_1)}, \quad (13)$$

and  $P_2^*$  is the solution to

$$\begin{aligned} &(\beta_1 - \beta_2) B_2 (P_2^*)^{\beta_2} + (\beta_1 - 1) P_2^* / \delta \\ &\quad - \beta_1 (C/r + I_2) = 0. \end{aligned} \quad (14)$$

Eq. (14) must be solved numerically for  $P_2^*$ .

# Investment Rule for a Two-Stage Project (con't.)

- Solution given by eq. (12) applies for  $P < P_2^*$ . When  $P \geq P_2^*$  the firm exercises option to invest, and  $F_2(P) = V(P) - I_2$ :

$$F_2(P) = \begin{cases} D_2 P^{\beta_1} & \text{for } P < P_2^* \\ V(P) - I_2 & \text{for } P \geq P_2^* \end{cases} \quad (15)$$

# Investment Rule for a Two-Stage Project (con't.)

- First-Stage Investment. Given  $F_2(P)$  and  $P_2^*$ , can back up to first stage and find value of option to invest,  $F_1(P)$ , and critical price  $P_1^*$ .
  - $F_1(P)$  also satisfies eq. (8), but now subject to

$$F_1(0) = 0 \quad (16)$$

$$F_1(P_1^*) = F_2(P_1^*) - I_1 \quad (17)$$

$$F_1'(P_1^*) = F_2'(P_1^*) \quad (18)$$

- Solution has the usual form:

$$F_1(P) = D_1 P^{\beta_1}. \quad (19)$$

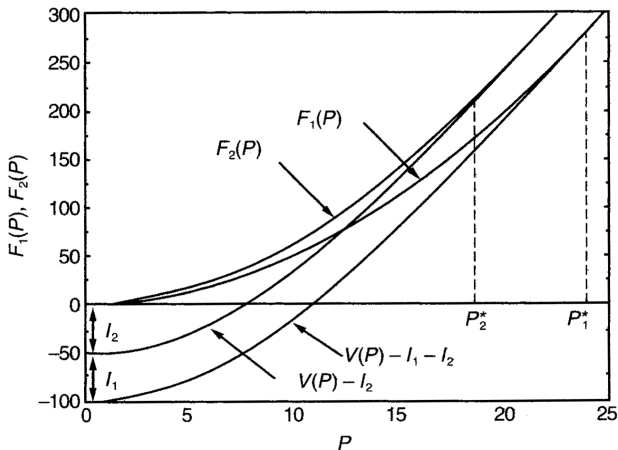
Use (17) and (18) to find  $D_1$  and critical price  $P_1^*$ . Because  $P_1^* > P_2^*$ ,  $F_2(P_1^*) = V(P_1^*) - I_2$ .

# Investment Rule for a Two-Stage Project (con't.)

- Since  $P_1^* > P_2^*$ , and since the investment can be completed instantaneously (before  $P$  can change), we know that once  $P$  reaches  $P_1^*$  and the firm invests, *it will complete both stages of the project*. This result seems anti-climactic, so why bother to solve this two-stage problem? Why not simply combine the two stages?
  - First, real-world investing takes time, so firms often do complete early stages and then wait before proceeding with later stages.
  - Second, the two stages may require different technical or managerial skills, may be located in different countries, or may be subject to different tax treatments. For these reasons one firm may sell a partially completed project to another. Our method lets us value a partially completed project.
  - Third, we will apply this approach to problems where completion of the project takes time.



# Critical Prices and Option Values for Two-Stage Project



# Investment Rule for a Two-Stage Project (con't.)

- Extension to projects with three or more stages: Start at the end and work backwards, using solution for each stage in the boundary conditions for the previous stage. Value of option to invest in any stage  $j$  of an  $N$ -stage project is of the form

$$F_j(P) = D_j P^{\beta_1},$$

and the coefficient  $D_j$  and critical price  $P_j^*$  are found by solving equations (13) and (14), with  $P_j^*$  replacing  $P_2^*$  and  $I_j + I_{j+1} + \dots + I_N$  replacing  $I_2$ .

# Introduction to Jump Processes

- Sometimes uncertainty is discrete in nature.
  - Competitor enters with better product, making yours worthless.
  - New regulations make your factory worth less (or more).
  - Sudden, unexpected success in the laboratory.
  - Foreign operation is expropriated, or tax treatment changed.
  - War, financial collapse, pestilence, etc.
- As long as these discrete events are non-systematic (diversifiable), easy to handle.
- Model as *jump* (Poisson) process,  $dq$ . Analogous to Wiener process:

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ u & \text{with probability } \lambda dt. \end{cases}$$

where  $u$  is the size of the jump (and can be random).

# Simple Example: Value of a Machine

- Suppose a machine produces constant flow of profit,  $\pi$ , as long as it operates.
- First, assume it lasts forever and never fails. No risk. Then asset return equation is:

$$rVdt = \pi dt$$

and value of machine is  $V = \pi/r$ .

## Simple Example: Value of a Machine (con't.)

- Now suppose at some point machine will break down and have to be discarded. So value of the machine follows the process:

$$dV - Vdq$$

where  $dq$  is a *jump* (Poisson) process. Now asset return equation is:

$$rVdt = \pi dt + \mathcal{E}(dV) = \pi dt - \lambda Vdt$$

Thus,

$$V = \frac{\pi}{r + \lambda}.$$

So just increase discount rate by  $\lambda$ .

# Undeveloped Oil Reserve

- Back to undeveloped oil reserve. Recall that value of developed reserve followed the process:

$$dV = (\mu - \delta)Vdt + \sigma Vdz$$

where  $\delta$  is payout rate net of depletion:

$$\delta = \omega(\Pi - V)/V \approx .04$$

- Now suppose a developed reserve is subject to full or partial expropriation. Then  $V$  follows:

$$dV = (\mu - \delta)Vdt + \sigma Vdz - Vdq$$

where  $dq$  is a jump process with mean arrival rate  $\lambda$ , and  $\mathcal{E}(dzdq) = 0$ .

# Undeveloped Oil Reserve (con't.)

- If “event” occurs,  $q$  falls by a fixed percentage  $\phi$  (with  $0 \leq \phi \leq 1$ ). Thus  $V$  fluctuates as a GBM, but over each  $dt$  there is a small probability  $\lambda dt$  that it will drop to  $(1 - \phi)$  times its original value, and then continue fluctuating until another event occurs.

# How to Estimate Arrival Rate $\lambda$ ?

- Begin with estimate of expected time  $T$  for event to occur, e.g., 5 years.
- Now get equation for  $\mathcal{E}(T)$ . Probability that no event occurs over  $(0, T)$  is  $e^{-\lambda T}$ . So probability that the first event occurs in the short interval  $(T, T + dT)$  is  $e^{-\lambda T} \lambda dT$ . So expected time until  $V$  jumps is:

$$\mathcal{E}[T] = \int_0^{\infty} \lambda T e^{-\lambda T} dT = 1/\lambda$$

- So if expected  $T$  is 5 years (60 months), use 0.2 (.01667) for  $\lambda$ .



# Optimal Investment Rule

- Want to find  $F(V)$ , value of undeveloped reserve, and optimal exercise point  $V^*$ .
- As we saw earlier, the  $dz$  component of  $dV$  can be “replicated.” We assume  $dq$  is non-systematic, i.e., can be diversified. So use risk-free rate. Then return equation is:

$$rFdt = \mathcal{E}(dF).$$

Expand  $dF$ :

$$r F dt = (\mu - \delta) V F'(V) dt + \frac{1}{2} \sigma^2 V^2 F''(V) dt - \lambda \{ F(V) - F[(1 - \phi) V] \} dt .$$

Can rewrite this as:

$$\frac{1}{2} \sigma^2 V^2 F''(V) + (r - \delta) V F'(V) - (r + \lambda) F(V) + \lambda F[(1 - \phi) V] = 0 .$$

The same boundary conditions apply as before.

# Optimal Investment Rule

- Solution is again of the form  $F(V) = AV^{\beta_1}$ , but now  $\beta_1$  is the positive solution to a slightly more complicated equation:

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - (r + \lambda) + \lambda(1 - \phi)^\beta = 0$$

Value of  $\beta$  that satisfies this and also satisfies  $F(0) = 0$  can be found numerically. Then  $V^*$  and  $A$  can be found.

- If  $\phi = 1$  (so “event” is that  $V$  falls to zero) above equation is a quadratic equation, and positive solution is:

$$\beta_1 = \frac{1}{2} - (r - \delta)/\sigma^2 + \sqrt{\left[(r - \delta)/\sigma^2 - \frac{1}{2}\right]^2 + 2(r + \lambda)/\sigma^2}$$

# Optimal Investment Rule (con't.)

- Table shows  $\beta_1$ ,  $V^*$ , and  $a$  for various values of  $\lambda$ , for case of  $\phi = 1$ . A positive value of  $\lambda$  affects  $F(V)$  in two ways.
  - First, it reduces the expected rate of capital gain on  $V$  (from  $\alpha$  to  $\alpha - \lambda$ ), which reduces  $F(V)$ .
  - Second, it increases variance of changes in  $V$ , which increases  $F(V)$ .
  - As the Table shows, *net* effect is to reduce  $F(V)$ , and thus reduce the critical value  $V^*$ .
  - Net effect is strong; small increases in  $\lambda$  lead to big drop in  $V^*$ .

# Dependence of $\beta_1$ , $V^*$ , and $A$ on $\lambda$

(Note:  $l = 1$ ,  $\phi = 1$ ,  $r = \delta = .04$ , and  $\sigma = .2$ .)

$\lambda$	$\beta$	$V^*$	$A$
0	2.00	2.00	.250
.05	2.70	1.59	.169
.1	3.19	1.46	.138
.2	4.00	1.33	.105
.3	4.65	1.27	.009
.5	5.72	1.21	.007
1.0	7.73	1.15	.005

# Optimal Investment Rule (continued)

- Note that we increased  $\lambda$  while holding  $\alpha = \mu - \delta$  fixed. Could argue that the market-determined expected rate of return on  $V$  should remain constant, so that an increase in  $\lambda$  is accompanied by a commensurate increase in  $\alpha$  (otherwise no investor would hold this project).
- Suppose  $\phi = 1$ . If  $\alpha$  increases as much as  $\lambda$  so  $\alpha - \lambda$  is constant, we have to replace the terms  $(r - \delta)$  in equation for  $\beta$  with  $(r + \lambda - \delta)$ . Then an increase in  $\lambda$  is like an increase in the risk-free rate  $r$ , and leads to an *increase* in  $F(V)$  and  $V^*$ .

# Optimal Investment Rule (continued)

- The simple jump process we used leads to a differential equation for  $F(V)$  that is easy to solve. Could specify different process for  $V$ .
  - Firm holding a patent faces competitors, each trying to develop its own patent. Success of a competitor might cause  $V$  to fall by a *random*, rather than fixed amount. Over time additional competitors may enter, so  $V$  continues to fall.
  - Calculation of optimal investment rule is more difficult, and would require numerical solution method.