# A Combinatorial Proof of Seymour's Conjecture for Regular Oriented Graphs with Regular Outsets $O'_a$ and $O''_a$

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**Abstract**

Let $G = (V, E)$ be a simple, oriented graph with minimum outdegree $= s$. Let $G^2$ be the digraph having the same vertex set, $V$, and an arc set: $E(G^2) = \{xy : xy \in E$ or $\exists w \in V[xw, wy \in E]\}$. According to Seymour's conjecture, there exists some vertex $a \in G$ whose outdegree in $G^2$ is at least twice that in $G$. The validity of this conjecture is proved for all regular $G$ with regular sets $O'_a$ and $O''_a$.

# 1 Introduction

## 1.1 Definitions and Terminology

A *graph*, $G$, is a structure composed of a set of points, $V$, known as *vertices* and the *edges*, $E$, connecting some subset[1] of pairs of these points. Any edge connecting two vertices is *adjacent* to those two vertices; two vertices joined by an edge are also adjacent. Here we consider *simple* graphs, where no more than one edge connects any two vertices and no loops[2] occur. A graph having every pair of vertices connected by an edge is called a *complete graph*. When every edge of a graph is directed by a pair of vertices we have a *directed graph* (digraph), and the edges are known as *arcs*. An *oriented graph* is a digraph having no bidirected arcs (symmetric pairs of directed arcs). A complete oriented graph is called a *tournament*[1].

The *outdegree* (*odeg*) of a vertex in a digraph is the number of arcs directed outward from that point; the *indegree* is the number of arcs directed inward. The *minimum outdegree* (*modeg*) of a graph or a set of vertices is the smallest outdegree held by any of the vertices. In a *regular graph* the outdegree of every vertex is equal; in an *almost regular graph* no two outdegrees differ by more than one[1]. The *outset* of a vertex $v$, denoted $O'_v$, is the set of all vertices to which an outward arc extends from $v$. The *second degree outset*, $O''_v$, may be defined as the union of the outsets of all $x$ in $O'_v$ minus $O'_v$.

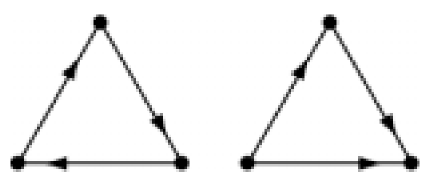

Figure 1: Examples of Oriented Graphs [1]

[1] This subset may be the null set.
[2] A degenerate edge connecting a vertex to itself

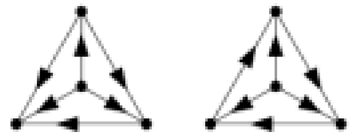

Figure 2: Examples of Tournaments [1]

### 1.2 Seymour's Conjecture

In 1993 Seymour formulated a conjecture concerning the square of an oriented graph, $G = (V, E)$. The *square* of $G$, $G^2$, is the digraph having the same vertex set, $V$, and an arc set, $E(G^2)$ containing all arcs $ab \in E$ as well as the arc $ac$ for every pair of arcs, $ab$ and $bc \in E$. The conjecture proposed by Seymour stated, "every oriented graph $G$ has a vertex whose outdegree in $G^2$ is at least twice its outdegree in $G$." A team of two mathematicians, Dean and Latka, were able to prove its validity for regular tournaments, almost regular tournaments, and tournaments with minimum outdegree $\leq 5$ in 1995. In 1996, Fisher was able to extend the proof to hold for all tournaments[2].

### 1.3 Proof Statement

For any regular oriented graph, $G = (V, E)$, with regular sets $O'_a$ and $O''_a$ for some $a \in V$, there exists some vertex whose outdegree at least doubles in $G^2$.

## 2 Lemmata

Several lemmata will be essential to the main proof. These principles are proved in the Appendix.

**Lemma 1** *Among $n$ vertices on a simple, oriented graph, the maximum number of arcs connecting these vertices is $\binom{n}{2}$.*

**Lemma 2** *Let $G = (V, E)$ be a regular oriented graph with $|V| = n$. Let $X = x_1, x_2, \dots x_s$ with $odeg(x_i) = s$ for all $i$. For $x_i, x_j \in X$ and $y \notin X$ let a repeat arc be $x_j y$ such that $x_i y$ also exists for $j > i$. The sum of the number of arcs of the form $ab$ for $a, b \in X$ and the number of repeat arcs has a maximum value of $s^2 - \lceil \frac{s+1}{2} \rceil$.*

**Lemma 3** *Let $G = (V, E)$ be an almost regular tournament with $|V| = s$. $|O''_{v_i}| \geq \lfloor \frac{s-1}{2} \rfloor$ for all $v_i \in V$.*

**Lemma 4** *Let $Y = (V, E)$ be an almost regular tournament with $|V| = n$ vertices and $|E| = \frac{n(n-1)}{2}$. If $v \notin Y$ is outwardly adjacent to $\lfloor \frac{s+1}{2} \rfloor$ of these vertices and $\lceil \frac{s+1}{2} \rceil \leq n \leq s-1$, then the number of vertices in $Y$ also in the second degree outset of $v$ is $n - \lfloor \frac{s+1}{2} \rfloor$.*

# 3 The Proof

## 3.1 An Overview

We attempt to validate, via proof by contradiction, Seymour's Conjecture for regular oriented graphs in which some vertex has a regular outset and a regular second degree outset. We first choose an arbitrary graph $G = (V, E)$ in obeyance of these conditions and let $modeg(G) = s$. We assume that Seymour's Conjecture does not hold. We then consider a vertex $a \in G$ with regular outset and regular second degree outset and establish the range of orders for its second degree outset, $O''_a$. The order of $O''_a$ is equivalent to the number of new arcs that will be outwardly directed from $a \in G^2$ and must be limited so that $odeg(a)$ does not at least double. This places the maximum order of $O''_a$ at $s - 1$. We note that the smallest order of $O''_a$ occurs when the sum of the number of arcs of the form $x_i x_j$ for $x_i, x_j \in O'_a$ and the number of *repeat arcs* is maximized. Thus, $\lceil \frac{s+1}{2} \rceil \leq |O''_a| \leq s - 1$.

To establish a contradiction, we next consider the second degree outsets of vertices $x_i \in O'_a$. Potential second degree outlets for some $x_i$ include vertices in $O'_a$ (*x-outlets*), vertices in $O''_a$ (*y-outlets*) and vertices not in $O'_a, O''_a$ (*w-outlets*). We necessarily minimize the sum of second degree outlets, as $x_i$ may not double in $G^2$. However, we find that the minimal sum of second degree outlets for some $x_1 \in O''_a$ is greater than or equal to $s$. Thus, $odeg(x_1)$ doubles in $G^2$. This is a contradiction. The initial set is the null set, and Seymour's Conjecture is validated for all combinations of $s$ and $n$.

### 3.2 The Outset

Let $G = (V, E)$ be a regular oriented graph with some vertex, $a$, having a regular outset and regular second degree outset. Assume there is no vertex in $G$ whose outdegree at least doubles in $G^2$. Let $s = modeg(G)$.

As $s = modeg(G)$, $odeg(a) = s$. Let $\{x_1, x_2, \ldots x_s\}$ be the outset, $O'_a$. For each $x_i$, $s$ outwardly directed arcs exist in $E$. As $s$ such $x_i$ exist, there is a total of $s^2$ arcs directed outward from $O'_a$. We define $Y$ as the set of all arcs of the form $x_i y \in E$. $Y$ is the sum of the outsets of all $x_i$; $|Y| = s^2$. The second degree outset, $O''_a$, is a subset of $Y$.

### 3.3 The Second Degree Outset of $a$

The order of the second degree outset of $a$ is equivalent to the number of new arcs $ay$ formed in $G^2$. Thus, to prevent the doubling of $odeg(a)$ in $G^2$, we must limit the size of $O''_a$. We consider two new factors in determining this size: the number of *repeats* and the number of *x-pairings* present in $Y$.

The set $O''_a$ was defined as the union of the outsets of all $x_i$ minus $O'_a$. The union of the outsets of $x_i$ is obtained by subtracting all *repeat arcs* from $Y$. A repeat arc is $x_j y \in Y$ when $x_i y$ exists for $x_i, x_j \in O'_a$, $1 \leq i, j \leq s$, and $j > i$ for some vertex $y$. Only the arc $x_i y$ need be counted when adding the arc $ay$ in $G^2$. Similarly, no vertex $x_j$ may be counted in $O''_a$ when $x_i x_j \in Y$, as $x_i, x_j \in O'_a$. These *x-pairings* are eliminated by subtracting $O'_a$.

As shown in Lemma 2, these two quantities have a maximum sum of $s^2 - \lceil \frac{s+1}{2} \rceil$. The minimum sum occurs when both terms are 0. The order of $O''_a$ can now be established as the order of Y ($= s^2$) minus this sum. This provides the following range: $s^2 - (s^2 - \lceil \frac{s+1}{2} \rceil) \leq |O''_a| \leq s^2$. However, there is a better upper bound, as $odeg(a)$ must not double in $G^2$. This restriction implies that $|O''_a| < |O'_a|$ and therefore $|O''_a| < s$. If not, the total $odeg(a) \in G$ would be greater than or equal to $2s$, contradicting our assumption. We can now examine a more limited range: $\lceil \frac{s+1}{2} \rceil \leq |O''_a| \leq s - 1$.

### 3.4 Second Degree Outlets for $x_1$

We now seek to enumerate the second degree outlets, vertices $v \in O''_{x_1}$ for some $x_1 \in O'_a$. There are three potential sources for such outlets: vertices $x_j \in O'_a$, $j \neq 1$ (*x-outlets*), vertices $y_k \in O''_a$ (*y-outlets*), and vertices $w_t \notin O'_a, O''_a$ (*w-outlets*). To ensure that $odeg(x_1)$ does not double in $G^2$, we seek to minimize the total number of second degree outlets.

We first consider the w-outlets. Each w-outlet must be inwardly adjacent to some $y_j \in O''_a$, as $w_t \notin O'_a, O''_a$. The number of arcs from $O''_a$ adjacent to $w_t \notin O'_a, O''_a$ ($|\vec{yw}|$) is limited by the number of arcs outwardly directed from $O''_a$ into $O''_a$ ($|\vec{yy}|$) and from $O''_a$ into $O'_a$ ($|\vec{yx}|$). As there are a total of $s|O''_a|$ arcs outwardly directed from $O''_a$,

$$|\vec{yw}| = s|O''_a| - |\vec{yy}| - |\vec{yx}|. \tag{1}$$

If $|\vec{yw}|$ arcs are distributed outward from $|O''_a|$ vertices, there exists at least one vertex, $y_1 \in O''_a$ outwardly adjacent to at least $\lceil \frac{|\vec{yw}|}{|O''_a|} \rceil$ w-outlets. We choose our $x_1$ such that $x_1 y_1 \in E$.

We see from Equation 1 that the number of w-outlets is minimized by an increase in the number of arcs adjoining two vertices within $O''_a$. For $|O''_a|$ vertices, $|\vec{yy}|$ is maximized at

$\frac{|O''_a|(|O''_a|-1)}{2}$ (Lemma 2). By substitution,

$$|\vec{yw}| = s|O''_a| - \frac{|O''_a|(|O''_a|-1)}{2} - |\vec{yx}|. \tag{2}$$

An increase in the number of arcs directed outwardly from $O''_a$ into $O'_a$ is also favorable in minimizing w-outlets. This increase is equivalent to minimizing the number of arcs directed outwardly from $O'_a$ into $O''_a$ or maximizing the number of arcs directed from $O'_a$ into $O'_a$ ($|\vec{xx}|$). There are a total of $s|O''_a|$ arcs that may be directed between the sets, $O'_a$ and $O''_a$. For a given number of arcs directed outwardly from $O'_a$ into $O''_a$ ($|\vec{xy}|$), there are a maximum of $s|O''_a| - |\vec{xy}|$ arcs directed from $O''_a$ into $O'_a$. The terms $|\vec{xy}|$ and $|\vec{xx}|$ are codependent as they must sum to $s^2$, the total number of arcs directed outward from $O'_a$. Thus, $|\vec{xy}| = s^2 - |\vec{xx}|$. Substituting, $|\vec{yx}| = s|O''_a| - (s^2 - |\vec{xx}|)$. As per Lemma 1, the term $|\vec{xx}|$ may be maximized at $\frac{s(s-1)}{2}$. This gives $|\vec{yx}| = s|O''_a| - (s^2 - \frac{s(s-1)}{2}) = s|O''_a| - s^2 + \frac{s(s-1)}{2}$. Substituting into Equation 2 and distributing negatives, we have

$$|\vec{yw}| = s|O''_a| - \frac{|O''_a|(|O''_a|-1)}{2} - s|O''_a| + s^2 - \frac{s(s-1)}{2}.$$

Certain terms cancel such that,

$$|\vec{yw}| = s^2 - \frac{s(s-1)}{2} - \frac{|O''_a|(|O''_a|-1)}{2}.$$

The number of y-outlets and x-outlets in the second degree outset of $x_1$ now immediately follow. Lemma 4 tells us that the total number arcs gained by $x_1$ in $G^2$ of the form $xy_l$ such that $xy_k$ and $y_ky_l$ exist for $y_k, y_l \in O''_a$ is $|O''_a| - \lfloor\frac{s+1}{2}\rfloor$. The total number of y-outlets in the second degree outset of $x_1$ must therefore be $|O''_a| - \lfloor\frac{s+1}{2}\rfloor$. Likewise, Lemma 3 tells us that for the regular subgraph of $O'_a$ with $\frac{s(s-1)}{2}$, the total number of x-outlets in the second degree outset of $x_i$ is at least $\lfloor\frac{s-1}{2}\rfloor$.

### 3.5 The Contradiction

Having enumerated the minimal combination of magnitudes of x-outlets, y-outlets, and w-outlets for $x_1 \in O'_a$, we now sum the three quantities and compare this value to $s$. The sum of the three quantities is equivalent to the order of the second degree outset of $x_1$, whereas $s$ is the order of the outset of $x_1$. If $|O''_{x_1}| \geq |O'_{x_1}|$, then $x_1$ must double in $G^2$. Thus,

$$\lceil \frac{|\vec{yw}|}{|O''_a|} \rceil + |O''_a| - \lfloor \frac{s+1}{2} \rfloor + \lfloor \frac{s-1}{2} \rfloor < s$$

$$\lceil \frac{s^2 - \frac{s(s-1)}{2} - \frac{|O''_a|(|O''_a|-1)}{2}}{|O''_a|} \rceil + |O''_a| - \lfloor \frac{s+1}{2} \rfloor + \lfloor \frac{s-1}{2} \rfloor < s$$

$$\lceil \frac{s^2+s}{2|O''_a|} \rceil - \frac{|O''_a|-1}{2} + |O''_a| - 1 < s$$

$$\lceil \frac{s^2+s}{2|O''_a|} \rceil + \frac{|O''_a|-1}{2} < s$$

$$\lceil \frac{s^2+s}{2|O''_a|} \rceil + \frac{|O''_a|-1}{2} - s < 0$$

$$\frac{s^2+s}{2|O''_a|} + \frac{|O''_a|-1}{2} - s < 0$$

$$s^2 + s + (|O''_a|-1)|O''_a| - 2|O''_a|s < 0$$

$$|O''_a|^2 - (2s+1)|O''_a| + s^2 + s < 0.$$

This is a quadratic in $|O''_a|$, so we apply the quadratic formula to solve for the zeroes:

$$|O''_a| = \frac{(2s+1) \pm \sqrt{(2s+1)^2 - 4(s^2+s)}}{2}$$

$$|O''_a| = \frac{(2s+1) \pm \sqrt{1}}{2}$$

$$|O''_a| = \frac{2s+1 \pm 1}{2}.$$

As the quadratic defines a concave up parabola, the inequality $|O''_a|^2-(2s+1)|O''_a|+s^2+s<0$ holds when $|O''_a|$ is between its roots. Thus, $x_1$ doubles when

$$\frac{2s+1-1}{2} \leq |O''_a| \leq \frac{2s+1+1}{2}$$

$$s \leq |O''_a| \leq s+1.$$

We know that $|O''_a| \leq s-1$. This is a contradiction. $|O''_{x_1}| \geq |O'_{x_1}|$, and our original assumption is invalid. Thus, Seymour's Conjecture must hold for all oriented graphs $G = (V, E)$ with regular outsets $O'_a$ and $O''_a$.

# 4 Conclusion

We have applied a unique combinatorial method to second degree outset size in order to demonstrate the validity of Seymour's conjecture for all regular oriented graphs with regular $O'_a$ and $O''_a$ for some vertex $a$. By contradiction, we have proved that, for any such regular oriented graph $G = (V, E)$ there does indeed exist some vertex in $G$ whose outdegree will at least double in $G^2$. Further work on this problem could lead to the formulation of algorithm able to determine the precise vertex whose outdegree must double in each case and perhaps reveal the presence of two vertices whose outdegrees must at least double in regular oriented graphs. Moreover, an expansion of this method considering the combination of minimum outdegree and the number of vertices of any oriented graph shows much potential in proving Seymour's Conjecture in its full generality.

# 5 Acknowledgments

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# References


[1] E. W. Weisstein. Eric Weisstein's World of Mathematics. CRC Oress LLC (1999) at http://mathworld.wolfram.com/.

[2] N. Dean. *Degrees and Digraphs*. (2001) at http://caam.rice.edu/~nated/caam470/notes/degrees.ps.

[3] N. Dean. *The Square of a Directed Graph*. Open Problems for Undergraduates. at http://dimacs.rutgers.edu/~hochberg/undopen/graphtheory/graphtheory.html.

# A Proof of Lemma 1

There are $n(n-1)$ ways to choose two vertices from $n$ vertices to form an arc. However, a simple graph has no bidirection: both $xy$ and $yx$ cannot exist. Only $\frac{n(n-1)}{2}$ can be counted. This quantity is $\binom{n}{2}$.

# B Proof of Lemma 2

We first note that the two aforementioned quantities, that of repeat arcs and that of $ab$ pairings, are codependent. The total number of arcs outdirected from X is limited to a value of $s^2$. Thus, the more arcs attributed to one quantity, the less left to be attributed to the other. We allow $r$ to represent the number of repeats and $p$ to represent the number of $ab$ pairings and seek to determine the maximum number of repeats that may exist for a given number of pairings in a formulaic manner.

We next note that the number of repeats is equivalent to the number of instances of intersection or overlap among outsets of the members of $X$. Maximum overlap can occur when the differences among sizes of these sets is minimized. Thus, to obtain maximum overlap, we look at a distribution of $p$ $ab$ pairings over $s$ vertices that minimizes the differences in number of outwardly directed arcs assigned to each vertex. This also minimizes the difference in the number of arcs per vertex that must be adjacent to some $y \notin X$. A distribution in which the number of outward $ab$ pairing arcs for any two vertices differ by no more than one is achieved by invoking simple division. Divide the $p$ pairings by the $s$ number of vertices and apply a number of outward arcs equivalent to the quotient of the division to each arc ($\lfloor \frac{p}{s} \rfloor$). Then, distribute the remainder so that no vertex receives more than one new arc assignment.

Following the assignment of $ab$ pairings, there remain $s^2 - p$ arcs open to initial instances and repeats of the form $x_i y$ for $x \in X, y \notin X$. For a given number of arcs, a smaller number of initial instances signifies a greater number of repeats. The smallest number of initial instances is determined by the largest number of outward arcs adjacent to some $x \in X$ not yet assigned inwardly to some $y$. Following the above distribution, the minimum number of outward $ab$ pairing arcs assigned to any vertex was $\lfloor \frac{p}{s} \rfloor$. Thus, the maximum number of unassigned arcs outwardly directed from any vertex is $s - \lfloor \frac{p}{s} \rfloor$, and this quantity is equal to the minimum number of initial instances for distinct $y \notin X$. All remaining arcs to the order of $s^2 - p - (s - \lfloor \frac{p}{s} \rfloor)$ may be considered to be repeats.

Thus, the maximum value for $r$ for a given $p$ is $s^2 - p - s + \lfloor \frac{p}{s} \rfloor$, but the maximum value of $r + p$ is needed. As the terms $p$ and $-p$ cancel in this summation, the only remaining factor to be considered is $\lfloor \frac{p}{s} \rfloor$. This factor increases as $p$ is maximized. The maximum value of $p$ among $s$ vertices is $\binom{s}{2}$ or $\frac{s(s-1)}{2}$ as proved in Lemma 1. Substituting this value, $r+p = s^2 - s + \lfloor \frac{\frac{s(s-1)}{2}}{s} \rfloor = s^2 - (s - \lfloor \frac{(s-1)}{2} \rfloor)$. If $s$ is odd, the difference $s - \lfloor \frac{(s-1)}{2} \rfloor = s - \frac{(s-1)}{2} = \frac{(s+1)}{2} = \lceil \frac{(s+1)}{2} \rceil$. If $s$ is even, the difference $s - \lfloor \frac{(s-1)}{2} \rfloor = s - \frac{(s-2)}{2} = \frac{(s+2)}{2} = \lceil \frac{(s+1)}{2} \rceil$. In final substitution, the maximum value of $r + p = s^2 - \lceil \frac{(s+1)}{2} \rceil$.

# C Proof of Lemma 3

Case 1: $s$ is odd

In the odd case, there are a total of $\frac{s(s-1)}{2}$ outwardly directed arcs (Lemma 1) and $s$ vertices. Thus there are $\frac{s-1}{2}$ vertices directed outward from each vertex. Some arc connects every two vertices. If we choose an arbitrary $v_1 \in V$, $|O'_{v_1}| = \frac{s-1}{2}$. The remaining $\frac{s-1}{2}$ vertices must be outwardly adjacent to $v_1$. We assume there is some vertex not in $O'_{v_1}$ which is also not in $O''_{v_1}$. This vertex is not inwardly adjacent to any vertex in $O'_{v_1}$ and thus must be outwardly adjacent to all vertices in $O'_{v_1}$. This vertex thus has an outdegree of $\frac{s-1}{2} + 1$, which is a contradiction. Therefore, $|O''_{v_1}| = \frac{s-1}{2}$. Without loss of generality, this holds for all $v_i$ such that $|O''_{v_i}| = \frac{s-1}{2}$.

Case 2: $s$ is even

In the even case, there are a total of $\frac{s(s-1)}{2}$ outwardly directed arcs (Lemma 1) and $s$ vertices. Thus there are an average of $\frac{s-1}{2}$ vertices directed outward from each vertex. This value is not an integer; therefore half of all vertices have an outdegree of $\frac{s-2}{2}$ and half of $\frac{s}{2}$. Some arc connects every two vertices. If we choose an arbitrary $v_1 \in V$, $|O'_{v_1}| = \frac{s}{2}$. The remaining $\frac{s-2}{2}$ vertices must be outwardly adjacent to $v_1$. We assume there is some vertex not in $O'_{v_1}$ which is also not in $O''_{v_1}$. This vertex is not inwardly adjacent to any vertex in $O'_{v_1}$ and thus must be outwardly adjacent to all vertices in $O'_{v_1}$. This vertex thus has an outdegree of $\frac{s}{2}+1$, which is a contradiction. Therefore, $|O''_{v_1}| = \frac{s-2}{2}$. Without loss of generality, this holds for all $v_i$ with $odeg(v_i) = \frac{s}{2}$ such that $|O''_{v_i}| = \frac{s-2}{2}$.

If we choose an arbitrary $v_1 \in V$, $|O'_{v_1}| = \frac{s-2}{2}$. The remaining $\frac{s}{2}$ vertices must be outwardly adjacent to $v_1$. We assume there are two vertices, $v_2, v_3$ not in $O'_{v_1}$ which are also not in $O''_{v_1}$. The vertex $v_2$ is not inwardly adjacent to any vertex in $O'_{v_1}$ and thus must be outwardly adjacent to all vertices in $O'_{v_1}$. This vertex thus has an outdegree of $\frac{s-2}{2}+1 = \frac{s}{2}$. This vertex is not outwardly adjacent to $v_3$; thus $v_3$ is outwardly adjacent to $v_2$ as well as to all vertices in $O'_{v_1}$. The $odeg(v_3) = \frac{s-2}{2}+2$, which is a contradiction. Therefore, $|O''_{v_1}| \geq \frac{s-2}{2}$. Without loss of generality, this holds for all $v_i$ with $odeg(v_i) = \frac{s-2}{2}$ such that $|O''_{v_i}| \geq \frac{s-2}{2} = \lfloor\frac{s-1}{2}\rfloor$.

# D Proof of Lemma 4

The vertex $v_0$ is outwardly adjacent to $\lfloor\frac{s+1}{2}\rfloor$ vertices in $Y$. Assume that there is one vertex in $Y$ not in $O'_{v_0}$ which is not in $O''_{v_0}$. This vertex must be outwardly adjacent to the $\lfloor\frac{s+1}{2}\rfloor$ vertices in $O'_{v_0}$. However, the outdegree of every vertex in $Y$ is at most $\lceil\frac{n-1}{2}\rceil$. Since, $s-1 \geq n$, $\frac{s}{2} \geq \frac{n+1}{2}$ and $\lfloor\frac{s+1}{2}\rfloor > \lceil\frac{n-1}{2}\rceil$. This is a contradiction. Thus, all vertices in $Y$ not in $O'_{v_0}$ are in $O''_{v_0}$. $|O''_{v_0}| =$ n-$\lfloor\frac{s+1}{2}\rfloor$.