A Partial Characterization of Ehrenfeucht-Fraïssé Games on Fields and Vector Spaces

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Abstract

In this paper we examine Ehrenfeucht-Fraïssé (EF) games on fields and vector spaces. We find the exact length of the EF game on two algebraically closed fields of finite transcendence degree over $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$. We also determine an upper bound on the length of any EF game on models $(\mathbb{F}_1^n, \mathbb{F}_1)$ and $(\mathbb{F}_2^m, \mathbb{F}_2$ of vector spaces where $m \neq n$ and a lower bound for the special case $\mathbb{F}_1 = \mathbb{F}_2$.

1 Introduction

Ehrenfeucht-Fraïssé (EF) games provide a numerical measure of the degree to which two models[1] are similar. They have applications in various branches of mathematical logic, such as measuring the expressive strength of a formal language. We begin with an example to see how EF games work and how they connect to other measures of similarity between models.

Suppose we wish to compare $\mathbb{Q}$ and $\mathbb{Z}$. They are clearly different, but they do share some properties. Consider attempting to distinguish between finite subsets of $\mathbb{Q}$ and $\mathbb{Z}$. In the language $\mathfrak{L} = \{<\}$, it is impossible to distinguish between $\{\frac{22}{7}, 1, \frac{3}{2}, 1.01\}^{\mathbb{Q}}$ and $\{21, 10, 20, 15\}^{\mathbb{Z}}$. That is, for any atomic formula φ in $\mathfrak{L}$, $\varphi(\frac{22}{7}, 1, \frac{3}{2}, 1.01)$ iff $\varphi(21, 10, 20, 15)$.

We call the map $\{\frac{22}{7}^{\mathbb{Q}} \to 21^{\mathbb{Z}}, 1^{\mathbb{Q}} \to 10^{\mathbb{Z}}, \frac{3}{2}^{\mathbb{Q}} \to 20^{\mathbb{Z}}, 1.01^{\mathbb{Q}} \to 15^{\mathbb{Z}}\}$, denoted $\{\frac{22}{7}, 1, \frac{3}{2}, 1.01\}^{\mathbb{Q}} \Longrightarrow \{21, 10, 20, 15\}^{\mathbb{Z}}$, a partial isomorphism from $\mathbb{Q}$ to $\mathbb{Z}$ because it maps a finite subset of $\mathbb{Q}$ to a finite subset of $\mathbb{Z}$, is bijective, and preserves structure. (For a formal definition and more examples of partial isomorphisms, see Appendix C.)

Now we ask whether it is possible to extend this partial isomorphism. If we added the element $3^{\mathbb{Q}}$, for instance, could we find some element $x^{\mathbb{Z}}$ to make $\{\frac{22}{7}, 1, \frac{3}{2}, 1.01, 3\}^{\mathbb{Q}} \Longrightarrow \{21, 10, 20, 15, x\}^{\mathbb{Z}}$ a partial isomorphism? Here the answer is no, because if $3^{\mathbb{Q}}$ is between $\frac{3}{2}^{\mathbb{Q}}$ and $\frac{22}{7}^{\mathbb{Q}}$, then $x^{\mathbb{Z}}$ must likewise be between $20^{\mathbb{Z}}$ and $21^{\mathbb{Z}}$, but there is no integer between 20 and 21.

We simulate these extensions of partial isomorphisms with an EF game on two models:

Definition 1.1. *[3, 5] In an Ehrenfeucht-Fraïssé (EF) game on $(M, \mathfrak{L})$ and $(N, \mathfrak{L})$ (the "M vs. N" EF game) where M and N are models of the language $\mathfrak{L}$, two players Spoiler and Duplicator create a partial isomorphism from M to N in stages as follows:*

At stage i, beginning with stage 1,

1. *Spoiler "plays" an element m_i^M or n_i^N by appending it to the list $\bar{m}^M$ or $\bar{n}^N$ of elements*

[1] For a glossary of terms which may be unfamiliar, see Appendix A.

played in that model.

2. *Duplicator "plays" an element from the other model, n_i^N or m_i^M.*

After each stage, we examine the mapping $f : \bar{m} \to \bar{n}$ which maps m_j to n_j for all $j \leq i$. If f is not a partial isomorphism, i.e. there exists some atomic formula in $\mathfrak{L}$ which is true of $\bar{m}$ but not of $\bar{n}$, then Spoiler wins the game at stage i. If f is a partial isomorphism, then Duplicator wins stage i. Duplicator wins the game if he has a strategy to win through any stage, i.e. to prevent Spoiler from ever winning.[2]

We return to our original example. In the EF game on $(\mathbb{Q}, \{<\})$ and $(\mathbb{Z}, \{<\})$, Spoiler is able to win by Stage 3 by the following strategy:

1. Spoiler plays two consecutive integers in $\mathbb{Z}$. Duplicator plays two corresponding elements $q_1^{\mathbb{Q}}$ and $q_2^{\mathbb{Q}}$.

2. Spoiler plays $\left(\frac{q_1+q_2}{2}\right)^{\mathbb{Q}}$, which is between q_1 and q_2. Since Duplicator cannot find an element in $\mathbb{Z}$ between the two consecutive integers Spoiler first played, Spoiler wins at stage 3.

Spoiler	5	$\in$	$\mathbb{Z}$	6	$\in$	$\mathbb{Z}$	4.6	$\in$	$\mathbb{Q}$
Duplicator	4.3	$\in$	$\mathbb{Q}$	4.9	$\in$	$\mathbb{Q}$	??	$\in$	$\mathbb{Z}$

Table 1: An EF game on $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$.

The reason such a strategy works can be described by a formula φ in $\mathfrak{L}$ such that $\mathbb{Q} \models \varphi$ and $\mathbb{Z} \models \neg\varphi$:

$$\varphi = \forall x \, \forall y \, \exists z \; \big((x < z) \wedge (z < y)\big) \vee \big((y < z) \wedge (z < x)\big) \vee (x = y)$$

[2] For 0 examples of EF games, see Appendix H.

In English, φ says that there exists a third element between any two nonequal elements. Because this is true in $\mathbb{Q}$ but not in $\mathbb{Z}$, Spoiler has a winning strategy. Note that the quantifier rank of φ is 3, the number of moves required for Spoiler to win. This correlation is true generally:

Theorem 1.2. *[1] Duplicator wins the EF game on $(M, \mathfrak{L})$ and $(N, \mathfrak{L})$ through stage i iff M and N are equivalent under formulas of quantifier rank i in $\mathfrak{L}_{\infty\omega}$.*[3]

We will also use sequences of partial isomorphisms in proving bounds on the lengths of EF games. A sequence of partial isomorphisms from M to N consists of

$$I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n$$

where each I_i is a collection of partial isomorphisms collectively mapping all of M to all of N, and any $p \in I_i$ has an extension in I_{i-1} for any element in M or N. (A formal definition and examples may be found in Appendix D.)

Theorem 1.3. *[3] There exists a sequence of partial isomorphisms from M to N of length i iff Spoiler cannot win the M vs. N EF game by stage i.*

In this paper, we consider EF games over fields and vector spaces. For various conditions on two such models M and N, we construct partial isomorphism sequences between M and N to place lower bounds on the stage at which Spoiler can win the M vs. N EF game, and we formulate strategies for Spoiler to put upper bounds on this stage.

[3]See Appendix E.

2 Preliminary Definitions

In order to play EF games on models of fields[4] or vector spaces[5], we must first define $\mathfrak{L}_F$, a language for fields, and $\mathfrak{L}_{VS}$, a language for vector spaces.

Definition 2.1. *The language of fields, $\mathfrak{L}_F$, is the set of symbols $\{+, \cdot, -, ^{-1}, 0, 1\}$. The language of vector spaces, $\mathfrak{L}_{VS}$, is the set consisting of*

- $S(x)$ *a unary predicate that is true when x is a scalar*
- $V(x)$ *a unary predicate that is true when x is a vector*
- $+^s$ *scalar addition*
- $\cdot^s$ *scalar multiplication*
- 0^s *scalar zero (additive identity)*
- 1^s *scalar one (multiplicative identity)*
- $+^v$ *vector addition*
- 0^v *the zero vector*
- $*$ *scalar multiplication of a vector*

Now we define the *field EF game* to be an EF game on two models M , N of $\mathfrak{L}_F$ such that M,$N \models \mathrm{Th}\,(\text{fields})$. Similarly, we define the *vector space EF game* to be an EF game on two models M,N of $\mathfrak{L}_{VS}$ such that M,$N \models \mathrm{Th}\,(\text{vector spaces})$.

Let $\mathbb{F}$ denote a field and $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. A full list of notation which may be unfamiliar to the reader is included as Appendix B.

[4]See Appendix F.
[5]See Appendix G.

3 EF field games

Our goal in this section is to characterize, as completely as possible, the general $\mathbb{F}_1$ vs. $\mathbb{F}_2$ field EF game. First we consider the case in which $\operatorname{char}(\mathbb{F}_1) \neq \operatorname{char}(\mathbb{F}_2)$:

Proposition 3.1. *Spoiler wins the $\mathbb{F}_1$ vs. $\mathbb{F}_2$ field EF game at stage 0 if and only if $\operatorname{char}(\mathbb{F}_1) \neq \operatorname{char}(\mathbb{F}_2)$. That is, no partial isomorphism sequence can be constructed between $\mathbb{F}_1$ and $\mathbb{F}_2$.*

Proof. If $\operatorname{char}(\mathbb{F}_1) \neq \operatorname{char}(\mathbb{F}_2)$, at least one of $\big(\operatorname{char}(\mathbb{F}_1), \operatorname{char}(\mathbb{F}_2)\big)$ is nonzero. Therefore, at least one of $\mathbb{F}_1, \mathbb{F}_2$ models an atomic formula in $\mathfrak{L}$ of the form $1+1+\ldots+1 = 0$. Since the characteristic of the other field is different, the other field does not model the same atomic formula, and Spoiler wins the EF game at stage 0.

If $\operatorname{char}(\mathbb{F}_1) = \operatorname{char}(\mathbb{F}_2)$, there is no atomic formula φ in $\mathfrak{L}_{\infty\omega}$ with only constants 0 and 1 such that $\mathbb{F}_1 \models \varphi$, $\mathbb{F}_2 \models \neg\varphi$, so Spoiler cannot win at stage 0. Thus, Spoiler wins at stage 0 if and only if the characteristics of the fields are unequal. $\square$

For the rest of the paper we assume that all fields are extensions of $\mathbf{Q}$, where $\mathbf{Q}$ is either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$. That is, all fields are of the same characteristic.

3.1 Fields of different transcendence degrees/$\mathbb{F}$

Next, we consider two fields $\mathbb{E}_1$, $\mathbb{E}_2$ extending $\mathbb{F}$ with different transcendence degrees over $\mathbb{F}$. In this case, we show that Spoiler wins the $\mathbb{E}_1$ vs. $\mathbb{E}_2$ field EF game by some finite stage:

Proposition 3.2. *If $\mathbb{E}_1$ and $\mathbb{E}_2$ extend $\mathbb{F}$, $\mathbb{E}_1$ has finite transcendence degree $a/\mathbb{F}$, $\mathbb{E}_2$ has transcendence degree b/F, and $a < b$, then Spoiler wins the $\mathbb{E}_1$ vs. $\mathbb{E}_2$ field EF game.*

Proof. First note that any atomic formula in $\mathfrak{L}_F$ can be written $f(x_1, \ldots, x_n) = 0$ for some $f \in \mathbf{Q}[x_1, \ldots, x_n]$.

Spoiler's strategy is to first play $c_1, \ldots, c_{a+1} \in \mathbb{E}_2$ such that $\{c_1, \ldots, c_{a+1}\}$ is algebraically independent over $\mathbb{F}$.

Duplicator plays $d_1, \ldots, d_{a+1} \in \mathbb{E}_1$. Since the transcendence degree of $\mathbb{E}_1$ is a, only a set of up to a elements in $\mathbb{E}_1$ can be algebraically independent over $\mathbb{F}_1$, and the set $\{d_1, \ldots, d_{a+1}\}$ is algebraically dependent over $\mathbb{F}$. This means that there exists a function $f(x_1, \ldots, x_{a+1}) \in \mathbb{F}[x_1, \ldots, x_{a+1}]$ such that $f(d_1, \ldots, d_{a+1}) = 0$. Because $\{c_1, \ldots, c_{a+1}\}$ is algebraically independent over $\mathbb{F}$, there does not exist a function $g(x_1, \ldots, x_{a+1}) \in \mathbb{F}[x_1, \ldots, x_{a+1}] : g(c_1, \ldots, c_{a+1}) = 0$. Thus, to win, Spoiler needs only to list every coefficient in f as an element in E_1. If there are i such coefficients $b_1, \ldots, b_i$, then there exists $h \in \mathbf{Q}[x_1, \ldots, x_{a+1+i}]$ such that $h(d_1, \ldots, d_{a+1}, b_1, \ldots, b_i) = 0$ iff $f(d_1, \ldots, d_{a+1}) = 0$. No corresponding elements $e_1, \ldots, e_i \in \mathbb{E}_2$ satisfy $h(c_1, \ldots, c_{a+1}, e_1, \ldots, e_i) = 0$. Therefore, Spoiler wins at some finite stage $a + i + 1$. $\square$

3.2 The field EF game on two algebraically closed fields

Finally, we consider the case in which $\mathbb{F}_1$ and $\mathbb{F}_2$ are algebraically closed fields of equal characteristic and with finite transcendence degrees over $\mathbf{Q}$. In this case, we know exactly at what stage Spoiler wins the $\mathbb{F}_1$ vs. $\mathbb{F}_2$ field EF game, assuming he and Duplicator both employ optimal strategies. First, we need two lemmas:

Lemma 3.3. *If $\{a_1, \ldots, a_n\}$ is algebraically independent over $\mathbb{F}_1$, $\{b_1, \ldots, b_n\}$ is algebraically independent over $\mathbb{F}_2$, $\mathbb{F}_1 \cong \mathbb{F}_2$, $\mathbb{E}_1 = \mathbb{F}(a_1, \ldots, a_n)$, $\mathbb{E}_2 = \mathbb{F}(b_1, \ldots, b_n)$, then there exists an isomorphism from $\mathbb{E}_1$ to $\mathbb{E}_2$ extending an isomorphism from $\mathbb{F}_1$ to $\mathbb{F}_2$.*

Lemma 3.4. *If $\{a_1, \ldots, a_n\}$ is a transcendence base for $\mathbb{E}_1 = \overline{\mathbb{F}_1(a_1, \ldots, a_n)}$ over $\mathbb{F}_1$ and $\{b_1, \ldots, b_n\}$ is a transcendence base for $\mathbb{E}_2 = \overline{\mathbb{F}_2(b_1, \ldots, b_n)}$ over $\mathbb{F}_2$, $\mathbb{F}_1 \cong \mathbb{F}_2$ then $\mathbb{E}_1 \cong \mathbb{E}_2$.*

Sketch of the proof. We let $\mathbb{E}_1^0 = \mathbb{F}(a_1, \ldots, a_n)$ and $\mathbb{E}_2^0 = \mathbb{F}(b_1, \ldots, b_n)$. Because $\mathbb{E}_1^0$ and $\mathbb{E}_2^0$ are both the minimum extensions of $\mathbb{F}$containing a set of n elements transcendental over

$\mathbb{F}$, $\mathbb{E}_1^0 \cong \mathbb{E}_2^0$ by Lemma 3.3. This implies that there exists some isomorphism $i : \mathbb{E}_1^0 \to \mathbb{E}_2^0$. The isomorphism i can be extended to an isomorphism from $\overline{\mathbb{E}_1^0}$, the splitting field of $\mathbb{E}_1^0$ over all monic polynomials in $\mathbb{E}_1[x]$, to $\overline{\mathbb{E}_2^0}$, the splitting field of $\mathbb{E}_2^0$ over all monic polynomials in $\mathbb{E}_2[x]$, and $\mathbb{E}_1 = \overline{\mathbb{E}_1^0} \cong \overline{\mathbb{E}_2^0} = E_2$ [7].

Now we proceed to the EF game on two algebraically closed fields:

Theorem 3.5. *Spoiler wins the $\mathbb{F}_1 = \overline{\mathbf{Q}(u_1, \ldots, u_n)}$ vs. $\mathbb{F}_2 = \overline{\mathbf{Q}(v_1, \ldots, v_m)}$ field EF game where $m < n$ exactly at stage $m+1$ if $\{u_1, \ldots, u_n\}$ is a transcendence base for $\mathbb{F}_1/\mathbf{Q}$ and $\{v_1, \ldots, v_m\}$ is a transcendence base for $\mathbb{F}_2/\mathbf{Q}$.*

First we show that Spoiler can always win by stage $m+1$. Then we show that Duplicator has a winning strategy through stage m by constructing a sequence of partial isomorphisms of length m between $\mathbb{F}_1$ and $\mathbb{F}_2$.

Spoiler's strategy to win by stage $m+1$ is to play $u_1, \ldots, u_{m+1} \in \mathbb{F}_1$. These are algebraically independent over $\mathbf{Q}$, i.e. there does not exist a function $f(x_1, \ldots, x_{m+1}) \in \mathbf{Q}[x_1, \ldots, x_{m+1}]$ such that $f(u_1, \ldots, u_{m+1}) = 0$. Duplicator wins through stage m by playing m elements $c_1, \ldots, c_m \in \mathbb{F}_2$ such that $c_1, \ldots, c_m$ is algebraically independent over $\mathbf{Q}$. However, since these m elements form a transcendence base for $\mathbb{F}_2/\mathbf{Q}$, every element c_{m+1} is algebraic over $c_1, \ldots, c_m$. Therefore, there exists a function $g(x_1, \ldots, x_{m+1}) \in \mathbf{Q}[x_1, \ldots, x_{m+1}]$ such that $g(c_1, \ldots, c_{m+1}) = 0$. But $g(u_1, \ldots, u_{m+1}) \neq 0$, so Spoiler wins at stage $m+1$.

Now we show that Spoiler cannot win *before* stage $m+1$ by constructing a sequence of partial isomorphisms of length m from $\mathbb{F}_1$ to $\mathbb{F}_2$.

Recall that any two algebraically closed fields of the same transcendence degree over $\mathbf{Q}$ are isomorphic by Lemma 3.4. Let I^j be the set of all isomorphisms $f : \mathbb{F}_1' \subseteq \mathbb{F}_1 \to \mathbb{F}_2' \subseteq \mathbb{F}_2$ where $\mathbb{F}_1'$ and $\mathbb{F}_2'$ are algebraically closed and transcendence degree $\mathbb{F}_1'/\mathbf{Q} =$ transcendence degree $\mathbb{F}_2'/\mathbf{Q} = j$.

Then let $I_i = \{p \subset f, |\operatorname{dom}(p)| < \omega, f \in I^j, j \leq m - i\}$.

Claim: $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-1}$ is a sequence of partial isomorphisms.

Proof. First we must show that the collective domains and ranges of the partial isomorphisms in any I_i are the entire $\mathbb{F}_1$ and $\mathbb{F}_2$ respectively, i.e.:

$$\forall I_i \text{ and } a \in \mathbb{F}_1, \text{ there exist } p \in I_i \text{ and } b \in \mathbb{F}_2 \text{ such that } p(a) = b$$

$$\forall I_i \text{ and } a \in \mathbb{F}_2, \text{ there exist } p \in I_i \text{ and } a \in \mathbb{F}_1 \text{ such that } p(a) = b$$

Note that any $a \in \mathbb{F}_1$ is either transcendental or algebraic over $\mathbf{Q}$. Let $a \in \mathbb{F}_1$ be transcendental over $\mathbf{Q}$ and $a' \in \mathbb{F}_1$ be algebraic over $\mathbf{Q}$. We let $\mathbb{F}_1' = \overline{\mathbf{Q}(a)}$, which has transcendence degree 1 over $\mathbf{Q}$. There exists an isomorphism $f : \mathbb{F}_1' \to \mathbb{F}_2'$ in I^1 for some $\mathbb{F}_2'$ of transcendence degree 1 over $\mathbf{Q}$ and a partial isomorphism $p \subset f$ such that $\{a, a'\} \in \operatorname{dom}(p)$ and $|\operatorname{dom}(p)| < \omega$. For all $i \leq m-1$, p is in I_i. Thus, any element in $\mathbb{F}_1$ is in the domain of some partial isomorphism in I_i. Similarly, any element in $\mathbb{F}_2$ is in the image of some partial isomorphism in I_i.

Next, we must show that for any element a in $\mathbb{F}_1$ and any partial isomorphism p in $I_i (i \neq 0)$, there is an extension of p in I_{i-1} with a in the domain. Likewise, for any element b in $\mathbb{F}_2$ there must be any extension with b in the image, i.e.:

$$\forall p \in I_i \text{ and } a \in \mathbb{F}_1, \text{ there exist } p' \in I_{i-1} \text{ and } b \in \mathbb{F}_2 \text{ such that } p'(a) = b \tag{1}$$

$$\forall p \in I_i \text{ and } b \in \mathbb{F}_2, \text{ there exist } p' \in I_{i-1} \text{ and } a \in \mathbb{F}_1 \text{ such that } p'(a) = b \tag{2}$$

We let $p \in I_i (i \neq 0)$ be a subset of $f : \mathbb{F}_1' \to \mathbb{F}_2' \in I^j, j \leq m - i$.

To prove (1), we let a be in $\mathbb{F}_1$ and consider two cases:

Case 1: $\boldsymbol{a \in \mathbb{F}_1'}$ Let $p' \in I_{i-1}$ be a subset of f such that $p' \supseteq p$ and $a \in \operatorname{dom}(p')$.

Case 2: $\boldsymbol{a \notin \mathbb{F}_1'}$ In this case, a is transcendental over $\mathbb{F}_1'$, as $\mathbb{F}_1'$ is algebraically closed. If the transcendence degree of $\mathbb{F}_2'/\mathbf{Q}$ is less than m, which is true for all $i > 0$, then there exists $b \in \mathbb{F}_2$ such that b is algebraically independent over $\mathbb{F}_1'$. Let $\mathbb{F}_1'' = \overline{\mathbb{F}_1'(a)}$

and $\mathbb{F}_2'' = \overline{\mathbb{F}_2'(b)}$. By Lemma 3.4, there exists an isomorphism $f' : \mathbb{F}_1'' \to \mathbb{F}_2''$. This isomorphism is in I^{j+1}, and so a subset $p' \subseteq f$ such that $a \in \mathrm{dom}\,(p')$, $|dom(p')| < \omega$ is in I_{i-1}.

The proof of (2) is analogous, because the dimension of $\mathbb{F}_2/\mathbf{Q}$ is less than that of $\mathbb{F}_1/\mathbf{Q}$ and so for $b \in \mathbb{F}_2$, $b \notin \mathbb{F}_2'$ there always exists an element $e \in (F_1 - F_1')$.

Thus, $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-1}$ is a partial isomorphism sequence, and by Theorem 1.3 Spoiler cannot win the $\mathbb{F}_1$ vs. $\mathbb{F}_2$ field EF game by stage m. Since he has a strategy to win by stage $m+1$, he wins exactly at stage $m+1$. □

4 EF vector space games

Now we turn our attention to vector space EF games. First, note that if M is an m-dimensional vector space over $\mathbb{F}$, we can without loss of generality choose a basis for this vector space and consider the model M to be $(\mathbb{F}^m, \mathbb{F})$. (See Appendix G.)

Next, note that if Spoiler wins the $\mathbb{F}_1$ vs. $\mathbb{F}_2$ field EF game by stage t, he also wins the $M = (\mathbb{F}_1^m, \mathbb{F}_1)$ vs. $N = (\mathbb{F}_2^n, \mathbb{F}_2)$ vector space EF game by stage t. His strategy is the same as in the field EF game.

4.1 EF games on vector spaces of different dimension

We determine an upper bound on the length of a general vector space EF game on vector spaces of different dimensions:

Theorem 4.1. *If $M = (\mathbb{F}_1^m, \mathbb{F}_1)$ and $N = (\mathbb{F}_2^n, \mathbb{F}_2)$, $m < n$, then Spoiler wins the M vs. N vector space EF game by stage $m + 1 + \min\,(m, \mathrm{mingen}\,(\mathbb{F}_1/\mathbf{Q}))$.*

Proof. Spoiler's strategy to win by stage $m + 1 + \min\,(m, \mathrm{mingen}\,(\mathbb{F}_1/\mathbf{Q}))$ is as follows:

1. Play $m+1$ linearly independent vectors $v_1, \ldots, v_{m+1} \in N$. Duplicator plays $w_1, \ldots, w_{m+1} \in M$. Assume $w_1, \ldots, w_m$ are independent. Because any m independent vectors in $\mathbb{F}_1^m$ span $\mathbb{F}_1^m$, w_{m+1} can be written as:

$$w_{m+1} = c_1 w_1 + c_2 w_2 + \ldots + c_m w_m, c_i \in \mathbb{F}_1.$$

   We let $f(x_1, \ldots, x_m, y_1, \ldots, y_{m+1}) = x_1 y_1 + \ldots + x_m y_m - y_{m+1}$ for scalars $x_1, \ldots, x_m$ and vectors $y_1, \ldots, y_{m+1}$. Then $f(c_1, \ldots, c_m, w_1, \ldots, w_{m+1}) = 0$.

2. Play the smallest subset $S = \{s_1, \ldots, s_t\} \subseteq \{c_i : c_i \in \mathbb{F}_1 - \mathbf{Q}\}$ such that $\mathbf{Q}(S) = \mathbf{Q}(c_1, \ldots, c_m)$, i.e. every c_i is in $\mathbf{Q}(s_1, \ldots, s_t)$. The size of this subset, t, cannot be greater than $\text{mingen}\,(\mathbb{F}_1/\mathbf{Q})$, since $c_1, \ldots, c_m \in \mathbb{F}_1$. It also cannot be greater than m, since the set of all c_i has size m. Thus, at most, $t = \min\,(m, \text{mingen}\,(\mathbb{F}_1/\mathbf{Q}))$. Let $c_i = g_i(s_1, \ldots, s_t)$ for some $g_i(x_1, \ldots, x_t) \in \mathbf{Q}[x_1, \ldots, x_t]$. So $s_1, \ldots, s_t, w_1, \ldots, w_{m+1}$ satisfy

$$f(g_1(s_1, \ldots, s_t), g_2(s_1, \ldots, s_t), \ldots, g_m(s_1, \ldots, s_t), w_1, \ldots, w_{m+1}) = 0. \tag{3}$$

   Duplicator must choose $r_1, \ldots, r_t \in N$ such that

$$f(g_1(r_1, \ldots, r_t), g_2(r_1, \ldots, r_t), \ldots, g_m(r_1, \ldots, r_t), v_1, \ldots, v_{m+1}) = 0. \tag{4}$$

   But since $g_i(r_1, \ldots, r_t)$ is in $\mathbb{F}_2$, this implies that v_{m+1} is linearly dependent on $v_1, \ldots, v_m$. Since $v_1, \ldots, v_{m+1}$ are linearly independent, choosing $r_1, \ldots, r_t \in N$ to satisfy (4) is impossible. Spoiler thus wins by stage $m + 1 + t = m + 1 + \min\,(m, \text{mingen}\,(\mathbb{F}_1/\mathbf{Q}))$.

□

We now consider the special case where $\mathbb{F}_1 = \mathbb{F}_2$. In this case we prove a *lower* bound on the stage at which Spoiler can win.

4.2 Spoiler cannot win the $\mathbb{F}^n$ vs. $\mathbb{F}^m$ EF game by stage m

Theorem 4.2. *If $M = (\mathbb{F}^m, \mathbb{F})$ and $N = (\mathbb{F}^n, \mathbb{F})$ are m, $m < n$, then Spoiler cannot win the M vs. N vector space EF game by stage m.*

Proof. To show that Spoiler cannot win the N vs. M EF game by stage m, we construct a sequence of partial isomorphisms of length m from N to M. Recall that any two vector spaces of the same dimension over the same field are isomorphic. We let I^j be the set of all partial isomorphisms from a j-dimensional subspace of $\mathbb{F}^n$ to a j-dimensional subspace of $\mathbb{F}^m$:

$$I^j = \{f : \mathbf{S}_n \subseteq \mathbb{F}^n \to \mathbf{S}_m \subseteq \mathbb{F}^m, \dim(\mathbf{S}_n/\mathbb{F}) = \dim(\mathbf{S}_m/\mathbb{F}) = j\}$$

Then we consider the partial isomorphisms generated as finite subsets of the isomorphisms in I^j. We define I_i to be the set of all partial isomorphisms $p = p_1 \cup p_2$ where p_1 is a finite subset of some f in $I^j, j \leq m - i$ and p_2 is a finite subset of the identity function on $\mathbb{F}$.

We claim that $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-1}$ is a sequence of partial isomorphisms. To prove this we must show that it fits the three definitional criteria of a sequence of partial isomorphisms:

1. $\forall I_i$, $a \in N$, $b \in M$, there exist $p, p' \in I_i$ such that $a \in \mathrm{dom}\,(p)$, $b \in \mathrm{Im}\,(p')$

   To show this, first let c be a scalar in $\mathbb{F}$. The partial isomorphism $(c \to c) \subset \mathrm{Id}\,(\mathbb{F})$ is in every I_i. Thus, for any scalar $a \in N$, there exists $p \in I_i : a \in \mathrm{dom}\,(p)$. Likewise, for any scalar $b \in M$, there exists $p \in I_i : b \in \mathrm{Im}\,(p)$.

   Now let v be a vector in N. Since v is in the one-dimensional subspace spanned by v, there exist an isomorphism $f \in I^1$ which maps $\mathbf{S}_n \subseteq \mathbb{F}^n$ to $\mathbf{S}_m \subseteq \mathbb{F}^m$ such that $\dim\,(\mathbf{S}_n/\mathbb{F}) = \dim\,(\mathbf{S}_m/\mathbb{F}) = 1$ and a subset p of f such that $v \in \mathrm{dom}\,(p)$. Similarly, for any vector $v' \in M$, there exist an isomorphism $f' \in I^1$ mapping $\mathbf{S}'_n \subseteq \mathbb{F}^n$ to $\mathbf{S}'_m \subseteq \mathbb{F}^m$

such that $\dim(\mathbf{S}'_n/\mathbb{F}) = \dim(\mathbf{S}'_m/\mathbb{F}) = 1$ and a subset p' of f' such that $v' \in \mathrm{Im}\,(p')$. Since p and p' are in I^1, they are also in all I_i for $i < m$.

Thus, for any elements $a \in N$, $b \in M$ and any I_i in our sequence, there is a partial isomorphism in I_i mapping a to an element of M and a partial isomorphism mapping some element of N to b.

2. $\forall p \in I_i$ and $a \in N$ there exist $p' \in I_{i-1}$ and $b \in \mathbb{F}^m$ such that $p'(a) = b$

   Let $p = p_1 \cup p_2$ be in I_i with p_1 a subset of $f \in I^j$, $j \leq m - i$,

   Case 1: a is a scalar. Here we let $p' = p_1 \cup \big(p_2 \cup (a \to a)\big)$. This partial isomorphism is in $I_i \subseteq I_{i-1}$ and maps a to a.

   Case 2: a is a vector. There are two possibilities:

   Case 2a: $\boldsymbol{a \in}$ span (dom ($\boldsymbol{p_1}$)). Let $p' = p_1 \cup (a \to f(a)) \cup p_2$. This partial isomorphism is in $I_i \subseteq I_{i-1}$ and maps a to $f(a)$.

   Case 2b: $\boldsymbol{a \notin}$ span (dom ($\boldsymbol{p_1}$)). In this case, the dimension of $(\mathrm{dom}\,(p_1) \cup a)$ is $j+1$. If $j + 1 \leq m$, which must be true for $i > 0$ because $j \leq m - i$, then there exists a $(j+1)$-dimensional subspace $\Psi \subseteq \mathbb{F}^m$ containing $\mathrm{Im}\,(p_1)$. Let e in Ψ, be independent from $\mathrm{range}\,(p_1)$. Then let $f' : \mathrm{span}\,(\mathrm{dom}\,(p_1) \cup a) \to (\mathrm{range}\,(p_1) \cup e)$ be an isomorphism such that $f' \supset p_1$. Since $f' \in I^{j+1}$, $p' = p \cup (a \to f'(a))$ is in I_{i-1}.

   Together the two cases guarantee that there exists an extension of $p \in I_i$ in I_{i-1} for any $a \in N$.

3. $\forall p \in I_i$ and $b \in M$ there exist $p' \in I_{i-1}$ and $a \in N$ such that $p'(a) = b$

   As in step 2, we let b be in M and $p = p_1 \cup p_2$ be in I_i, $i > 0$, where $p_1 \subseteq f \in I^j, j \leq m - i$. The cases are analogous:

Case 1: b is a scalar. Here we let $p' = p_1 \cup \big(p_2 \cup (b \to b)\big)$. This partial isomorphism is in $I_i \subseteq I_{i-1}$ and maps b to b.

Case 2: b is a vector. Again we have two possibilities:

Case 2a: $\boldsymbol{b} \in$ span (Im ($\boldsymbol{p_1}$)). Here let $p' = p_1 \cup (f^{-1}(b) \to b) \cup p_2$. This partial isomorphism is in $I_i \subseteq I_{i-1}$ and maps $f^{-1}(b)$ to b.

Case 2b: $\boldsymbol{b} \notin$ span (Im ($\boldsymbol{p_1}$)). The dimension of $(\mathrm{Im}\,(p_1) \cup b)$ is $j+1$. Since $j+1 \leq n$, there exists a $(j+1)$-dimensional subspace $\Psi \subseteq \mathbb{F}^n$ containing $\mathrm{dom}\,(p)$. Let e be in Ψ and be independent from $\mathrm{dom}\,(p)$. Then let f' : $\mathrm{span}\,(\mathrm{dom}\,(p_1) \cup e) \to \mathrm{span}\,(\mathrm{Im}\,(p) \cup b)$ be an isomorphism such that $f' \supset p_1$. Since $f' \in I^{j+1}$, $p' = p \cup ((f')^{-1}(b) \to b)$ is in I_{i-1}.

Cases 1 and 2 show that for any $b \in F^m$ there is an extension $p' \in I_{i-1}$ of $p \in I_i$ with $b \in \mathrm{Im}\,(p')$.

Thus, $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-1}$ is a sequence of partial isomorphisms and, by Theorem 1.3, Spoiler cannot win the M vs. N vector space EF game by stage m. $\square$

5 Conclusion and Future Work

We have presented partial characterizations of the EF field game and EF vector space game, including the exact stage at which Spoiler wins an EF field game on algebraically closed fields of finite transcendence degree over $\mathbf{Q}$ and a minimal bound on the length of the EF vector space game on vector spaces of different dimension over the same field.

Remaining problems include:

1. The characterization of the general field EF game on two non algebraically closed fields of equal characteristic, $\mathbb{F}_1 \subseteq \overline{\mathbf{Q}(u_1, \ldots, u_n)}$ and $\mathbb{F}_2 \subseteq \overline{\mathbf{Q}(v_1, \ldots, v_m)}$.

2. The exact stage at which Spoiler wins an EF vector space game on vector spaces of dimensions n and m, $n < m$, over an algebraically closed field (conjecture: 2n+1).

3. An extension of the methods used here to the corresponding ring and module EF games.

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A Glossary

algebraic: An element a is algebraic over a set A if for some polynomial f with coefficients in A, $f(a) = 0$.

algebraic closure: The algebraic closure of a field $\mathbb{F}$ is the union of $\mathbb{F}$ and all elements algebraic over $\mathbb{F}$. This is a field, unique up to isomorphism[1].

algebraically dependent: A set S is algebraically dependent over a field $\mathbb{F}$ if $\exists a_1, \ldots, a_n \in S$ and $f \in \mathbb{F}[x_1, \ldots, x_n]$ such that $f(a_1, \ldots, a_n) = 0$.

arity: A function of arity n maps $M^n \to M$, i. e. takes n arguments. A relation of arity n takes n arguments. Example: $\varphi(x, y, z) \Leftrightarrow (x = y) \wedge (y = z)$ is a relation of arity 3.

atomic formula: A formula with no quantifiers.

bijective: Both injective and surjective, i.e. a one-to-one, onto map.

characteristic: The least positive p such that $1 + 1 + \ldots + 1$ (p times) $= 0$, or 0 if such an element does not exist.

equivalent under formulas of quantifier rank r: Models M and N are equivalent under formulas of quantifier rank r iff

$$\forall \varphi : \operatorname{qr}(\varphi) \leq r, \ \ M \models \varphi \Leftrightarrow N \models \varphi$$

field: See Appendix F.

formula: A statement in a language which can be either true or false. Formally, a formula in first-order logic is any of the following:

- $R(\tau_1 \ldots \tau_n)$

- $\tau_1 = \tau_2$
- $\varphi \wedge \Psi$
- $\neg\varphi$
- $\varphi \Rightarrow \Psi$
- $\varphi \vee \Psi$
- $\varphi \Leftrightarrow \Psi$
- $\exists x\varphi$
- $\forall x\varphi$

where R is a relation, $\tau_1 \ldots \tau_n$ are terms, and φ and Ψ are formulas.

free variable: A variable in a formula which is not mentioned in any quantifiers. For instance, in the formula $\forall x(y < x)$, y is a free variable.

homomorphism: If M and N are models of a theory in language $\mathfrak{L}$, a function $f : M \rightarrow N$ is a homomorphism if

$$\forall \text{ relations } R \in \mathfrak{L}, \bar{a} \in M, M \models R(\bar{a}) \text{ iff } N \models R(f(\bar{a}))$$

$$\forall \text{ functions } g \in \mathfrak{L}, \bar{a} \in M, M \models g(\bar{a}) = b \text{ iff } N \models g(f(\bar{a})) = f(b)$$

injective: If f is an injective function, then $f(a) = f(b) \Rightarrow a = b$, i.e. no two elements are mapped to the same value.

isomorphism: A homomorphism with an inverse, i.e. a bijective homomorphism.

language: A collection of constants, functions, and variables.

$\mathfrak{L}_1 = \{<\}$ and $\mathfrak{L}_2 = \{c_1, c_2, +, *, \leq\}$ are examples of languages.

An example of a formula "in" $\mathfrak{L}_2$ is $\forall x\big((x + c_1) * c_2 \leq x\big)$.

linearly independent: A set S is linearly independent if no element in S is a linear combination of the other elements.

model: A model M of a language $\mathfrak{L}$ is a set M with

- a subset $R^M \subseteq M^n$ for each relation $R \in \mathfrak{L}$ of arity n
- a function $f^M : M^n \to M$ for each function $f \in \mathfrak{L}$ of arity n
- a constant $c^M \in M$ for each constant $c \in \mathfrak{L}$

Basically, this is a "universe" in which a language has meaning and any sentence in the language is either true or false. Readers familiar with the programming language Java may find it helpful to think of a language as an interface and a model as a class implementing the interface.

In a model of a theory, every sentence in the theory is true. In a model of a sentence, the sentence is true.

module: A vector space over a ring instead of a field.

partial isomorphism: See Appendix C

partial isomorphism sequence: See Appendix D

quantifier: $\forall$ or $\exists$.

quantifier rank: The quantifier rank of a formula φ, written $\mathrm{qr}\,(\varphi)$, is defined inductively as follows:

$$\mathrm{qr}\,(\varphi) = 0 \quad \text{if } \varphi \text{ is atomic} \tag{5}$$

$$\mathrm{qr}\Big(\bigvee_{i\in I} \varphi_i\Big) = \mathrm{qr}\Big(\bigwedge_{i\in I} \varphi_i\Big) = \max\{\mathrm{qr}\,(\varphi_i) : i \in I\} \tag{6}$$

$$\mathrm{qr}\,(\neg\varphi) = \mathrm{qr}\,(\varphi) \tag{7}$$

$$\mathrm{qr}\,(\forall x\,\varphi) = \mathrm{qr}\,(\exists x\,\varphi) = \mathrm{qr}\,(\varphi) + 1 \tag{8}$$

$\mathrm{qr}\,(\varphi)$ is equal to the maximum number of nested quantifiers in φ.

ring: A set closed under two functions addition and multiplication, with multiplication distributive over addition. The integers, for instance, form a ring.

sentence: A formula with no free variables.

span: The span of a set of vectors $v_1, \ldots, v_n$ is the subspace consisting of all linear combinations of $v_1, \ldots, v_n$[10].

splitting field: The splitting field of a field $\mathbb{F}$ over a monic polynomial $f \in \mathbb{F}[x]$, where $f(x) = (x - r_1)(x - r_2)\ldots(x - r_n)$, is $\mathbb{F}(r_1, \ldots, r_n)$[6].

surjective: A map $f : M \to N$ is surjective iff $\forall n \in N \exists m \in M : f(m) = n$, i.e. $\mathrm{Im}\,(f) = N$.

term: A term is any of the following:

- a variable
- a constant
- the value of a function $f(\tau_1, \ldots, \tau_n)$, where $\tau_1, \ldots, \tau_n$ are terms and f has arity n.

theory: A collection of sentences.

transcendence base: If a field $\mathbb{E}$ is an extension of a field $\mathbb{F}$, $\mathbb{B} \subseteq \mathbb{E}$ is called a transcendence base of $\mathbb{E}/\mathbb{F}$ iff $\mathbb{B}$ is algebraically independent over $\mathbb{F}$ and every $a \in \mathbb{E}$ is algebraic over $\mathbb{F}(\mathbb{B})$. The size of such a transcendence base is unique[7].

transcendence degree of $\mathbb{E}/\mathbb{F}$: The size of the transcendence base of $\mathbb{E}/\mathbb{F}$.

transcendental: An element a is transcendental over a set A iff it is not the solution to any polynomial with coefficients in A.

vector space: See Appendix G.

B Notation

$\exists$	there exists
$\forall$	for all
$\neg$	not
$\Leftrightarrow$	if and only if
$\Rightarrow$	implies
ω	the size of the set of natural numbers
$\equiv^r$	equivalent under formulas of quantifier rank $\leq r$
$p' \supseteq p$	If p' and p are partial isomorphisms, $p' \supseteq p$ means that for $a \in \mathrm{dom}\,(p)$, $p'(a) = p(a)$, i.e. p' extends p.
a^M	denotes an element a in the model M
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of natural numbers
$\overline{\mathbb{F}}$	the algebraic closure of a field $\mathbb{F}$
$\mathbb{F}(u_1, \ldots, u_n)$	the smallest field which includes $\mathbb{F} \cup \{u_1, \ldots, u_n\}$
$\mathbb{F}[x_1, \ldots, x_n]$	the smallest ring which includes $\mathbb{F} \cup \{x_1, \ldots, x_n\}$, i.e. the polynomial ring with variables $x_1, \ldots, x_n$ and coefficients in $\mathbb{F}$
$\mathrm{mingen}\,(\mathbb{E}/\mathbb{F})$	the size of the smallest subset $\mathbf{S} \subseteq E$ such that $\mathbb{F}(\mathbf{S}) = \mathbb{E}$
$\mathrm{qr}\,(\varphi)$	the quantifier rank of φ
$\mathrm{Im}\,(f)$	the image of f
$\mathrm{dom}\,(f)$	the domain of f

C Partial isomorphisms

Definition C.1. *[1] A partial isomorphism is a bijective homomorphism $p : M \to N$ on a finite subset of M.*

We now provide some examples of partial isomorphisms:

1. The mapping $p : (\mathbb{Z}, \{<\}) \to (\mathbb{Q}, \{<\})$ which takes each element of $\{1, 4, 5, 3, 2\}^{\mathbb{Z}}$ to the corresponding element of $\{-4, 10.5, 30, \frac{5}{2}, 0\}^{\mathbb{Q}}$ is a partial isomorphism.

2. The mapping $p : (\mathbb{Q}(\pi), \mathfrak{L}_F) \to (\mathbb{Q}(e, \sqrt{2}), \mathfrak{L}_F)$ which takes each element of $\{\pi, \pi^2, \pi + 3\}^{\mathbb{Q}(\pi)}$ to the corresponding element of $\{e, e^2, e + 3\}^{\mathbb{Q}(e,\sqrt{2})}$ is a partial isomorphism. Note that there is no partial isomorphism which maps $\sqrt{2}^{\mathbb{Q}(e,\sqrt{2})}$ to any element x in $\mathbb{Q}(\pi)$, because x would have to satisfy the atomic formula $x * x = 1 + 1$ in $\mathfrak{L}_F$.

3. If $M = (\mathbb{R}^3, \mathbb{R})$ and $N = (\mathbb{R}^2, \mathbb{R})$, then $p : (M, \mathfrak{L}_{VS}) \to (N, \mathfrak{L}_{VS})$ which takes each element of $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}^M$ to the corresponding element of $\{(1, 0), (0, 1), (\pi, e)\}^N$ is a partial isomorphism. However, the mapping $p' \supseteq p$ which also takes π^M to π^N and e^M to e^N is not a partial isomorphism, because there is an atomic formula φ in $\mathfrak{L}_{VS}$ such that $\varphi((1, 0), (0, 1), (\pi, e), \pi, e)$ and $\neg\varphi((1, 0, 0), (0, 1, 0), (0, 0, 1), \pi, e)$:

$$\varphi(v_1, v_2, v_3, s_1, s_2) = v_1 * s_1 +^v v_1 * s_2 = v_3$$

D Sequences of partial isomorphisms

Definition D.1. *[1] $I_0 \supseteq I_1 \supseteq \ldots$ is a sequence of partial isomorphisms from M to N iff:*

1. *$\forall p \in I_i$, p is a partial isomorphism from M to N*

2. *$\forall I_i$ and $m \in M$ there exist $n \in N$, $p \in I_i : p(m) = n$;*

   $\forall I_i$ and $n \in N$ there exist $m \in M$, $p \in I_i : p(m) = n$

3. $\forall p \in I_i$ *and* $m \in M$ *there exist* $n \in N$ $p' \supseteq p$ *in* I_{i-1} *such that* $p'(m) = n$*;*
   $\forall p \in I_i$ *and* $n \in N$ *there exist* $m \in M$ *and* $p' \supseteq p$ *in* I_{i-1} *such that* $p'(m) = n$

In essence, this means that there are two main conditions on a sequence of partial isomorphisms: first, the collective domain and range of each I_i must be the entire M and N respectively; second, any p in I_i must be extendable to any given element of M or N.

An example is given in the figure below. This partial isomorphism sequence is between $(M, \{<\})$ and $(N, \{<\})$, where $M = \{1, 2, 3\}$ and $N = \{a, b, c, d\}, a < b < c < d$. Examining the figure, we see that for every every partial isomorphism p in I_1 we can choose any element in M or N and find an extension of p in I_0 containing a map to or from that element.

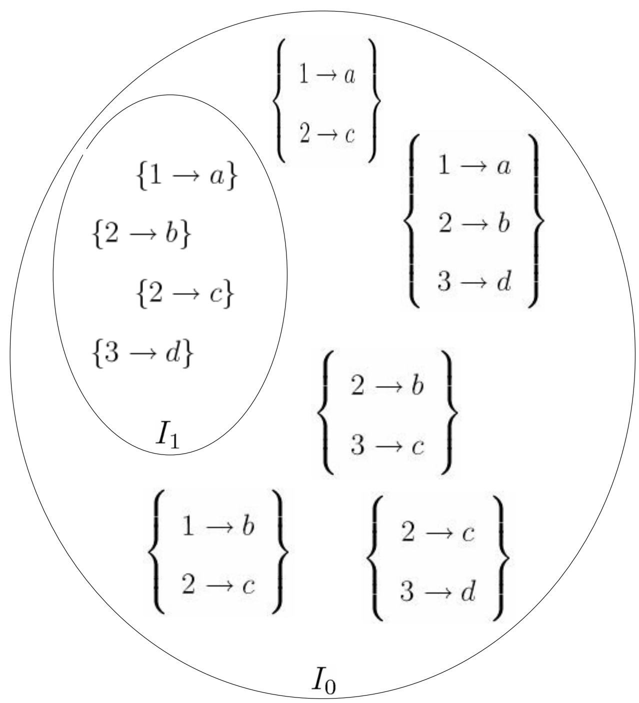

Figure 1: A sequence of partial isomorphisms of length 2 from M to N.

E $\mathfrak{L}_{\infty\omega}$

Theorem 1.2 states that Duplicator wins the EF game on $(M, \mathfrak{L})$ and $(N, \mathfrak{L})$ through stage i if and only if M and N are equivalent under formulas of quantifier rank $\leq i$ in $\mathfrak{L}_{\infty\omega}$.

$\mathfrak{L}_{\infty\omega}$ differs from $\mathfrak{L}$ in that infinite conjunctions and disjunctions are allowed, so long as the total number of free variables remains finite[2]. For example,

$$\varphi = \forall x \bigvee_{i \in \mathbb{N}} (x = i)$$

is a formula in $\mathfrak{L}_{\infty\omega}$. φ allows us to say that every x is a natural number. It is not a formula in $\mathfrak{L}$, because it involves taking an infinite conjunction, i.e. combining an infinite number of formulas with "or".

F Fields

Definition F.1. *A field is a model of $\mathfrak{L}_F$ which satisfies the field axioms below:*

Th (fields) is the set of the following axioms in $\mathfrak{L}_F$[11]:

Commutativity $\forall x, y(x + y = y + x) \wedge (x \cdot y = y \cdot x)$.

Associativity $\forall x, y, z((x + y) + z = x + (y + z)) \wedge (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$.

Distributivity$\forall x, y, z(x \cdot (y + z) = x \cdot y + y \cdot z)$.

Identity $\forall x(x + 0 = x) \wedge (x \cdot 1 = x)$.

Inverses $\forall x\big((x - x = 0) \wedge (x \neq 0 \Longrightarrow x \cdot x^{-1} = 1) \wedge (0^{-1} = 0)\big)$.

G Vector spaces

Definition G.1. *A vector space over a field* $\mathbb{F}$ *is a model of* $\mathfrak{L}_{VS}$ *which satisfies the vector space axioms (*$\mathrm{Th}\,($*vector spaces*$)$*) where* $\mathbb{F}$ *is considered the set of scalars.*

Th (vector spaces) is a set of sentences in $\mathfrak{L}_{VS}$ including $\forall x\big(S(x) \vee V(x)\big) \wedge \neg\big(S(x) \wedge V(x)\big)$ and

Axioms for scalars: Commutativity, associativity, distributivity, identity, and inverses, as in Th (fields), for elements which satisfy $S(x)$.

Axioms for vectors:

- $\forall x, y, z V(x) \wedge V(y) \wedge V(z) \iff x +^v (y +^v z) = (x +^v y) +^v z.$
- $\forall x, y V(x) \wedge V(y) \iff x +^v y = y +^v x.$
- $\forall x V(x) \Leftrightarrow x +^v 0^v = x.$
- $\forall x V(x) \Leftrightarrow \exists y : \big(V(y) \wedge x +^v y = 0^v\big)$

Axioms involving $*$:

- $\forall x V(x) \iff (0^s * x = 0^v) \wedge (1^s * x = x).$
- $\forall x, y, z S(x) \wedge V(y) \wedge V(z) \iff x * (y +^v z) = x * y +^v x * z.$
- $\forall x, y, z S(x) \wedge S(y) \wedge V(z) \iff ((x +^s y) * z = x * z +^v y * z) \wedge ((x \cdot y) * z = x * (y * z)).$
- $\forall x, y x * y = y * x.$

Axioms for nonsensical operations:

- $\forall x, y V(x) \implies (x \cdot y = 0^s) \wedge (x +^s y = 0^s).$
- $\forall x, y S(x) \implies x +^v y = 0^s.$

- $\forall x, y \big(V(x) \wedge V(y)\big) \vee \big(S(x) \wedge S(y)\big) \implies x * y = 0^s$.

Any two vector spaces of the same finite dimension over a field are isomorphic[9]. Also, for any finite-dimensional vector space there exists a basis. This allows us to consider an m-dimensional vector space over $\mathbb{F}$ as $\mathbb{F}^m$ without loss of generality.

H EF game examples

Here we provide some example EF games to help familiarize the reader with the strategy involved.

First, notice that Spoiler may list his plays in advance without gaining any advantage; for instance, he may "play" three elements at once and then wait for Duplicator to "catch up". For clarity, we may refer to Spoiler and Duplicator playing sets of elements at once, though in the actual game the elements are being played one at a time.

An example EF game is shown in Table 2 on $\{\mathbb{Z}, <\}$ and $\{\mathbb{Z} + \mathbb{Z}, <\}$. Here $\mathbb{Z} + \mathbb{Z}$ refers to two "copies" of $\mathbb{Z}$, one after another, such that any element in the second is larger than any in the first. To distinguish between integers in the two copies of $\mathbb{Z}$ in $\mathbb{Z} + \mathbb{Z}$, we will call an integer in the first copy small and an integer in the second copy big.

Spoiler wins this game by using the fact that there are infinitely many integers between any small integer and any big integer in $\mathbb{Z} + \mathbb{Z}$, but only a finite number of integers between any two elements of $\mathbb{Z}$. First he chooses big 0 and small 0 in $\mathbb{Z} + \mathbb{Z}$. Correspondingly, Duplicator chooses two elements of $\mathbb{Z}$, z_1 and z_2. Then Spoiler forces Duplicator to play integers between z_1 and z_2 until Duplicator must find an integer between consecutive integers, which is impossible. Using this strategy, Spoiler always wins, but Duplicator can determine how long it takes by setting the distance between z_1 and z_2. In the game depicted in Table 2, Spoiler wins at stage 7.

A second example is the $\mathbb{R}$ vs. $\mathbb{Q}(e)$ field EF game. In the example shown in Table 3,

Stage	Spoiler	Duplicator
1	small 0 $\in$ $\mathbb{Z}+\mathbb{Z}$	0 $\in$ $\mathbb{Z}$
2	big 0 $\in$ $\mathbb{Z}+\mathbb{Z}$	16 $\in$ $\mathbb{Z}$
3	small 10 $\in$ $\mathbb{Z}+\mathbb{Z}$	8 $\in$ $\mathbb{Z}$
4	small 20 $\in$ $\mathbb{Z}+\mathbb{Z}$	12 $\in$ $\mathbb{Z}$
5	small 100 $\in$ $\mathbb{Z}+\mathbb{Z}$	14 $\in$ $\mathbb{Z}$
6	big -10 $\in$ $\mathbb{Z}+\mathbb{Z}$	15 $\in$ $\mathbb{Z}$
7	small 200 $\in$ $\mathbb{Z}+\mathbb{Z}$	?? $\in$ $\mathbb{Z}$

Table 2: An EF game on $(\mathbb{Z},<)$ and $(\mathbb{Z}+\mathbb{Z},<)$.

Spoiler first plays $e^{\mathbb{R}}$. Duplicator plays $e^{\mathbb{Q}(e)}$. Then Spoiler plays $\pi^{\mathbb{R}}$. For any element $a^{\mathbb{Q}(e)}$ that Duplicator may play,

$$a = \frac{q_0 + q_1 e + \ldots + q_n e^n}{p_0 + p_1 e + \ldots + p_m e^m}$$

for some rational $q_0, \ldots, q_n, p_0, \ldots, p_m$. Then we have

$$a(p_0 + p_1 e + \ldots + p_m e^m) = q_0 + q_1 e + \ldots + q_n e^n \tag{9}$$

$$p_0 a + p_1 e a + p_2 e^2 a + \ldots + p_m e^m a - (q_0 + q_1 e + \ldots + q_n e^n) = 0 \tag{10}$$

which is equivalent to $f(e,a) = 0$ for a function $f \in \mathbf{Q}[x_1, x_2]$. But $f(e,\pi) \neq 0$, so Spoiler wins at stage 2.

Stage	Spoiler	Duplicator
1	e $\in$ $\mathbb{R}$	e $\in$ $\mathbb{Q}(e)$
2	π $\in$ $\mathbb{R}$	a $\in$ $\mathbb{Q}(e)$

Table 3: An EF field game on $\mathbb{R}$ and $\mathbb{Q}(e)$.

As a third example we take the M vs. N vector space EF game, where $M = (\mathbb{R}^2, \mathbb{R})$ and $N = (\mathbb{R}^3, \mathbb{R})$. A sample play is shown in Table 4. Spoiler begins by playing a basis for $\mathbb{R}^3$ in N. Duplicator plays the vectors $(1,0)$, $(0,1)$, and (π, e). Spoiler does not win yet, because there is no linear combination with coefficients in $\mathbf{Q}$ of the vectors $(1,0)$ and $(0,1)$ that yields (π, e).

Next, Spoiler plays π and e in M. Regardless of the field elements a and b that Duplicator chooses in N, there is an atomic formula φ in $\mathfrak{L}_{VS}$ such that $\{(1,0),(0,1),(\pi,e),\pi,e\}^M \models \varphi$ and $\{(1,0,0),(0,1,0),(0,0,1),a,b\}^N \models \neg\varphi$:

$$\varphi(v_1, v_2, v_3, s_1, s_2) = s_1 * v_1 +^v s_2 * v_2 = v_3$$

Thus, Spoiler wins at stage 5.

Stage	Spoiler			Duplicator		
1	$(1,0,0)$	$\in$	N	$(1,0)$	$\in$	M
2	$(0,1,0)$	$\in$	N	$(0,1)$	$\in$	M
3	$(0,0,1)$	$\in$	N	(π,e)	$\in$	M
4	π	$\in$	M	a	$\in$	N
5	e	$\in$	M	b	$\in$	N

Table 4: An EF vector space game between M and N.

For more examples and a more detailed discussion of the relationship between EF games and equivalence under formulas of a certain quantifier rank, see [3].