A relationship between Hochschild cohomology and the Goldman bracket for compact oriented two-dimensional manifolds.

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Abstract

It was shown in [3] that one can associate a certain infinite-dimensional Lie algebra to any two-dimensional oriented manifold S.

In this paper we relate Goldman's invariant to an algebraic construction, the Hochschild cohomology of the group algebra of the fundamental group, $HH^*(\mathbb{Q}(\pi_1(S)))$. We construct a Lie algebra homomorphism $\rho_1 : L \to HH^1(\mathbb{Q}(\pi_1(S))))$ and show that it is in fact an isomorphism with a certain natural ideal $H_0 \in HH^1(\mathbb{Q}(\pi_1(S)))$ of finite codimension. Thus we express the topological Goldman structure—something that can only be defined for two-dimensional oriented manifolds in terms of something more general.

1 Introduction

For any topological space S, one can define group structure on the set $\pi_1(S)$ of homotopy classes of *loops* (continuous functions from the circle S^1 to S) when both are viewed as pointed spaces (i.e., when there is a base point s_0 on S_1 and a base point x_0 in S, and all maps must take s_0 to x_0). If one forgets about the chosen points, it can be shown that the set F of homotopy classes of "free loops"—continuous maps $f : S^1 \to S$ without chosen points—is indexed by conjugacy classes of π_1. There is *a priori* no multiplication structure on this set. However, surprisingly, if S is an oriented two-dimensional manifold, one can define a Lie algebra structure on $\mathbb{Q}(F)$, the free vector space on F over $\mathbb{Q}$.

Let S be an oriented smooth two-dimensional manifold. When we have two free loops ϕ_1 and ϕ_2 which represent f_1 and f_2 in F, we cannot canonically choose a common point to multiply them as based loops. However, if the loops are "nice", we can multiply them in *every* common point and take the sum with signs over all of them.

First we define the "niceness" condition.

Definition 1. *We say that the a set of loops $\phi_1, \cdots, \phi_n$ is* generic *if all the loop maps $\phi_i : S^1 \to S$ are smooth and all intersections and self intersections of $\phi_1, \cdots, \phi_n$ are transversal.*

It is a standard result that up to homotopy, we can assume that these embeddings are generic up to homotopy.

For a generic pair of loops ϕ_1 and ϕ_2, we can define the sign of any intersection I of ϕ_1 with ϕ_2. We say $s(I) = \pm 1$ with $s(I) = 1$ when the angle between the tangent vector to ϕ_1 and the tangent vector to ϕ_2 is between 0 and π and $S(I) = -1$ otherwise.

If we choose an intersection I, we can also define maps ϕ_{1_I} and ϕ_{2_I} of *pointed* spaces $(S^1, s_0) \to (X, P)$: We view the circle as the group of unit complex numbers and set $\phi_{1_I}(z) =$

$\phi_{1_I}(zs_1^{-1})$ and $\phi_{2_I}(z) = \phi_{2_I}(zs_2^{-1})$. This allows us to multiply the loops as elements of the fundamental group, with the chosen point of X at I: we set $\phi_1 *_I \phi = \phi_{1_I} * \phi_{2_I}$ in this pointed space, but viewed as a free loop.

Definition 2. *The Lie bracket of two free loops can now be defined as a sum over all intersections I:*

$$[f_1, f_2] = \sum_I s(I)\phi_1 *_I \phi_2.$$

By linearity, this determines the product structure of the whole space.

This bracket is called the Goldman bracket and was first defined by the mathematician William Goldman in a 1986 paper on differential geometry ([3]). The bracket turned out to be important for the study of flat connections on two dimensional manifolds, as well as for other areas of differential geometry and physics.

In 2006, Chas and Sullivan described a generalization of the Goldman bracket to higher dimensional manifolds ([1]) by considering homology classes of the space of free loops instead of its connected components (called "string topology").

The algebra $\mathbb{Q}[C]$ with product structure defined by the Goldman bracket will be denoted L throughout this paper. It is not difficult to check that L is anticommutative and satisfies the Jacobi identity. It is not immediately obvious, however, that the Goldman product of two loops γ_1 and γ_2 depends only on the homotopy types of the loops. This is shown in [3] by proving that the bracket is conserved after applying a series of moves, similar to Reidemeister moves, which can get from any pair of loops γ_1, γ_2 with transversal intersections to any pair of homotopic loops.

Note that this definition depends on manifold structure and can not immediately be generalized to other spaces. One would like to relate the Goldman bracket to some more standard topological construction.

2 Definitions

Throughout the paper, S will be a compact two-dimensional oriented manifold with fundamental group $G = \pi_1(S)$. The letters g, g_1, g_2 denote elements of G represented by loops $\gamma, \gamma_1, \gamma_2$ respectively. Let F denote the set of conjugacy classes of G, which corresponds to *unbased* (free) loops on S. Throughout this paper, f, f_1, f_2 will denote elements of $[G]$ represented by loops ϕ, ϕ_1, ϕ_2 respectively.

Let $A = k(G)$ be the group algebra. a and b will denote arbitrary elements in A.

Definition 3. *The complex C We define C to be chain complex*

$$C^0 \to C^1 \to C^2 \to \cdots$$

where C^n is defined as $Hom_{\mathbb{Q}}(A^{\otimes n}, A)$ with differentials $d_n : Hom(A^{\otimes n}, M) \to Hom(A^{\otimes n+1}, M)$ defined by

$$d_n(f)(a_1 \otimes \cdots \otimes a_n) = a_1 f((a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$

$$+ f(a_1 \otimes \cdots \otimes a_{n-1}) a_n.$$

In particular, $d_1(a) \in Hom(A, A)$ is defined as $(d_1(a))(b) = ab - ba$.

Definition 4. *Let $B^i = Im(d^i)$ be the space of coboundaries and $Z^i = Ker(d_{i+1})$—the space of cocycles of C. We define $HH^n(A, M)$, the Hochschild cohomology of A, M for any A-bimodule M as the cohomology of the Hochschild chain complex, C: in other words the vector space quotient*

$$HH^i = H^i(C) = Z^i / B^i.$$

Remark 1. *In terms of homological algebra, $HH^n(A, M) = Ext^n(A, M)$ in the category of A-bimodules.*

Note that Z^1 is the space of maps $f : A \to A$ such that $0 = d(f)(g_1, g_2) = g_1 f(g_2) + f(g_1)g_2$, in other words the space of derivations of A. We can define a Lie-algebra bracket on the set Z^1 of derivations by the formula $[z_1, z_2] = z_1 \circ z_2 - z_2 \circ z_1$ for any $z_1, z_2 \in Z^1 \subset Hom(A, A)$. We see that B^1, also called the set of inner derivations, is an ideal in the Lie algebra Z^1. Thus the factor $HH^1 = Z^1/B^1$ inherits a Lie-algebra structure. This algebra is often called $Out(A)$, the space of "outer" derivations of A.

To relate L and HH^1, we will construct a map ρ from L to A and show that it is a Lie Algebra homomorphism.

3 Constructing ρ

We first define a map $\rho'(\phi')$ for any loop ϕ' which does not go through our chosen point x_0. Then we will show that this map is the same for any ρ'_1 which is homotopy equivalent to ρ'.

Let S' be the two-dimensional oriented manifold $S \backslash x_0$ (the complement in S to our base point). Let L' be the Goldman Lie Algebra on S'. Let ϕ' be a free loop, in S', representing $f' \in L'$. We define $\rho' : L' \to Hom(A, A)$ similarly to the Goldman bracket:

$$\rho'(f)(g) = \sum_{I \in \phi' \cup \gamma} s(I)[\phi' *_I \gamma],$$

where $[\gamma *_I \phi'] \in G$ represents $\gamma *_I \phi'$ viewed as a based loop. This map is well-defined, and this can be shown using the same arguments used in [3] to prove that the Goldman bracket is well-defined. We can see that $\rho(\phi')$ is a derivation because

$$\rho(\phi')(g_1 g_2) = \sum_{I \in \gamma_1 \cup \phi'} s(I)(\phi' *_I \gamma_1) *_{x_0} \gamma_2 + \sum_{I \in \gamma_2 \cup \phi'} s(I)\gamma_1 *_{x_0} (\phi' *_I \gamma_2) = \rho'(f)(g)h + g\rho(f)h.$$

Similarly, we see that the map $\rho' : L' \to Z^1$ is a Lie algebra homomorphism. We will show that it can be "pushed down" to a Lie algebra homomorphism $\rho : L \to HH^1$. The proof follows from the following lemma:

Lemma 1. *If ϕ'_1 and ϕ'_2 are two loops in S' homotopic as loops in S, then $\rho'(\phi'_1) - \rho'(\phi'_2) \in B^1$.*

Proof. Since ϕ'_1 and ϕ'_2 are homotopic on S, one can be obtained from the other by a finite sequence of homotopies in S' and moves where the loop crosses our chosen base point, as follows.

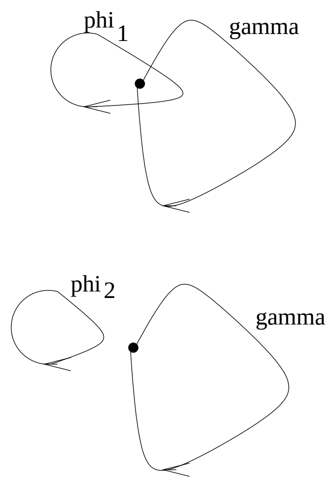


Figure 1: moving the loop through the base point

This means that it suffices to show that if ϕ'_1 and ϕ'_2 are related by such a move, then $\rho'(\phi'_1) - \rho'(\phi'_2) \in B^1$.

The difference $\rho'(\phi'_1) - \rho'(\phi'_2) \in B^1$ is equal to the difference of the two loops on the left in the following diagram (the components from all other intersections cancel). Now we see that if we define the *based* loop ϕ as in the following diagram (the based loop "between" ϕ_1

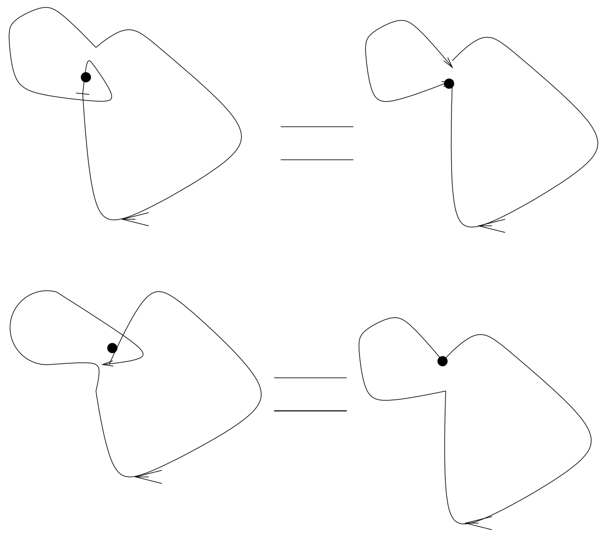


Figure 2: loops that do not cancel in $\rho'(\phi_1') - \rho'(\phi_2')$.

and ϕ_2) then the conjugation $[\phi * \gamma - \gamma * \phi]$ is represented by the difference of the two loops on the right in the above diagram. This means that $(\rho'(\phi_1') - \rho'(\phi_2'))(g) = [\phi]g - g[\phi]$. Thus

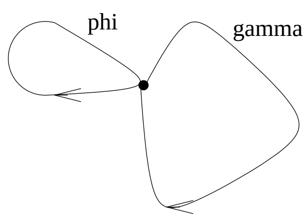


Figure 3: the based loop ϕ

$\rho'(\phi_1') - \rho'(\phi_2') \in B^1$ and we are done. $\square$

This allows us to define a Lie Algebra homomorphism $\rho : L \to Z^1/B^1 = HH^1$. We will now use the relationship between group cohomology and Hochschild cohomology to show that ρ is injective and has finite codimension.

4 Group cohomology

For any group G and left G-module M, we define a resolution

$$C_G^0 \to C_G^1 \to C_G^2 \to \cdots$$

where $C_G^n = Hom(A^{\otimes n}, M)$ for $n \geq 1$ and $C_G^0 = M$ where A is again the group algebra $\mathbb{Q}(G)$ with differentials $d_n : Hom(A^{\otimes n}, M) \to Hom(A^{\otimes n+1}, M)$ defined by

$$d_i(f)(a_1 \otimes \cdots \otimes a_n) = a_1 f((a_2 \otimes \cdots \otimes a_n) + \sum_{1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$

$$+(-1)^n f(a_1 \otimes \cdots \otimes a_{n-1}).$$

We define the *group* cohomology $H^i(G, M)$ as $H^i(C_G)$.

Remark 2. *In terms of Homological algebra, $H^i(G, M) = Ext^i_G(\mathbb{Q}, M)$ in the category of $\mathbb{Q}(G)$-modules where $\mathbb{Q}$ is regarded as a module over G with trivial action.*

Theorem 1. *Equivalence of resolutions Let G be any group and set $A = \mathbb{Q}(G)$. Then for any A-A bimodule M, the Hochschild coomology $HH^i(A, A)$ is isomorphic to the group homology $H^i(G, M)$, where G acts on M by conjugation (so we define the action $g.m$ of g on m is defined as gmg^{-1} (in A-module action).*

Proof. We construct a morphism of chain complexes. Let $\alpha : A^{\otimes n} \to A$ be a map in C^n. We define

$$T_* = \{t_1, t_2, \cdots\} : C \to C_G$$

where

$$t_n : \alpha \to t_n(\alpha) : g_1 \otimes \cdots \otimes g_n \mapsto \alpha(g_1 \otimes \cdots \otimes g_n) g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}$$

To check that these maps commute with differentials, we compute $(t_{n+1} \circ d_n)(\alpha)(g_1 \otimes \cdots \otimes$

$g_n) = g_1\alpha(g_2 \otimes \cdots \otimes g_n)g_n^{-1} \cdots g_1^{-1} + \sum_{i=1}^{n-1}(-1)^i\alpha(g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n)g_n^{-1} \cdots g_1{-}1 +$ $(-1)^n\alpha(g_1 \otimes \cdots \otimes g_{n-1} = d_n \circ f_n(\alpha)(g_1 \otimes \cdots \otimes g_n.$

Note that all the t_i involved in this map are isomorphisms. This means that the induced map on cohomology

$$T : HH^n(A, M) = H^n(C) \to H^n(C_G) = H^n(A, G)$$

is an isomorphism.

□

Let X be the conjugacy class corresponding to the free loop ϕ. G acts on X by conjugation, so we can define a left G-module $\mathbb{Q}(X)$. The G-module A (where G acts by conjugation) decomposes into a direct sum over conjugacy classes $\oplus_X \mathbb{Q}(X)$.

This means that C_G also decomposes into a direct sum over conjugacy classes:

$$C_G^n = \bigoplus_X C_X^n$$

where $C_X^n = Hom(A^{\otimes n}, \mathbb{Q}(X))$. It follows that

$$H^n(G, A) = \bigoplus_{X \in [G]} H^n(G, \mathbb{Q}(X)).$$

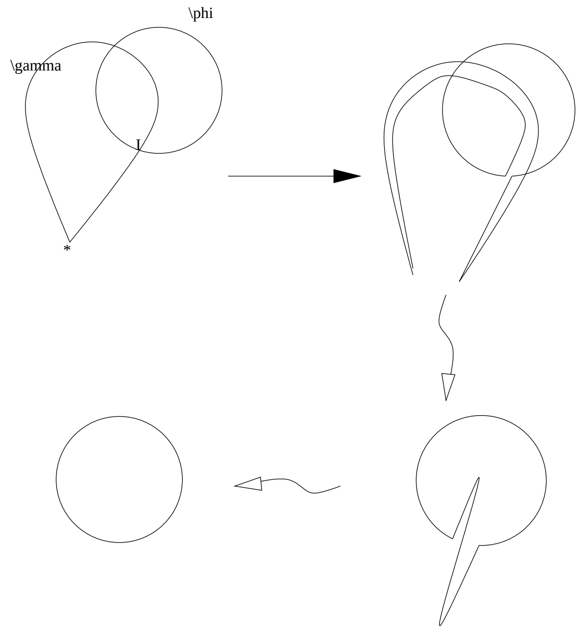
Lemma 2.

$$T(\rho(\phi)) \in H^1(\mathbb{Q}(X), G) \subset H^1(A, G).$$

Proof. We need to show that $T(\rho(f))(g) \in X$ for any homotopy class of free groups f. We know

$$T(\rho(f))(g) = \rho'(f)(g)g^{-1} = \sum_{I \in \phi' \cup \gamma} s(I)[\phi' *_I \gamma *_* \gamma^{-1}].$$

The following diagram shows that this is homotopic to γ as a free loop.



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Lemma 3. *For any $f_0 \neq f \in F$, $\rho(f) \neq 0$.*

Proof. We will use Etingof's result in [4] that the center of L is spanned by the trivial loop, f_0.

We define $p : A \to L$ to be the set surjection sending every element $g \in G$ which represents the loop γ to the unbased loop corresponding to γ. For any inner derivation $\alpha \in B^1(A)$, we can see that $p(\alpha(a)) = 0$ for any $a \in A$.

It is clear from the definition of ρ that $p(\rho'(f)(g)) = [f, p(g)]$.

Assume now that $\rho(f) = 0$. Then $p(\rho(f)(g)) = 0$ $\rho(f)(g) = 0$ for any $g \in G$ then $[f, p(g)] = 0$ for all G. But since p is a surjection, it follows that f is in the center of L.

Assume that $\rho(f) = 0$. This means that for any choice ϕ' of a loop in L' representing f and for any $g \in G$, we can choose $a \in A$ such that $\rho'(g) = ag - ga$. We can apply the natural projection $p : A \to L$ (which sends $g \in G$ to the conjugacy class of g) to both sides.

We get $p(ag - ga) = 0$ and $p(\rho'(f)(g)) = [f, p(g)]$ (since the definition of ρ' was the same as the definition of goldman bracket up to choice of base point).

Thus we have $[f, p(g)] = 0$ for any $g \in G$. Since $p : A \to L$ is injective, this means that $[f, f_1] = 0$ for any $f_1 \in F$. Therefore f is in the center of L. This is a contradiction by [4]. $\square$

Let f_0 be the generator of L which corresponds to the trivial loop. Then $\mathbb{Q}\gamma_0$ is in the center of L and $\rho(\gamma_0) = 0$. This means that we can define a Lie algebra hoomomorphism $\rho_0 : L/\mathbb{Q}\gamma_0 \to HH^1$.

We have proved that $\rho(f) \neq 0$ for any generator f of L except for f_0 and also that every f gets mapped to a different summand of HH^1 (as a vector space). This shows that ρ_0 is injective.

Theorem 2. *If X is a non-trivial conjugacy class of $G = \pi_1(S)$ then $H^0(\mathbb{Q}(X)) \cong 0$ $H^1(\mathbb{Q}(X)) \cong \mathbb{Q}$ and $H^2(\mathbb{Q}(X)) \cong \mathbb{Q}$.*

Proof. We know that G acts transitively on X. This means that if we define Z to be the the stabilizer of some element $x \in X$, then $\mathbb{Q}(X) \cong \mathbb{Q}(G/Z)$ as a left G-module. Note that $Z = Z(x)$ is the same as the centralizer in G of x.

Lemma 4. *For any element $1 \neq x \in G$, the centralizer $Z(x) \cong \mathbb{Z}$.*

Proof. Any closed compact two-dimensional surface of genus greater than one is a quotient of the upper half plane by a discrete group of hyperbolic transformations, so the fundamental group $G = \pi_1(S)$ is a discrete hyperbolic subgroup of $PSL_2(\mathbb{R})$.

It is a standard result in geometry that the centralizer of any hyperbolic element $1 \neq h \in PSL_2(\mathbb{R})$ is isomorphic to $\mathbb{R}$. Since G is a discrete subgroup, $Z(x)$ must be isomorphic to a discrete subgroup of $\mathbb{R}$, which can only be $\mathbb{Z}$ or 0. Since $1 \neq x \in Z(x)$, we know $Z(x)$ is non-trivial and $Z(x) \cong \mathbb{Z}$. $\square$

We fix an element $x \in X$ and set $H = Z(x)$. Let $M = \mathbb{Q}(G/H)$. We need to compute $HH^i(M) = \mathbb{Q}$. To do this, we relate this group cohomology to the cohomology with compact supports of the covering space of S corresponding to the subgroup H.

Let $\tilde{S}$ be the universal covering space of S. The fundamental group G acts simply transitively on $\tilde{S}$ so we can define a new oriented manifold $P = \tilde{S}/H$. By manipulating chain complexes, we can show that $H_c^i(P, \mathbb{Q})$, the cohomology of P with coefficients in $\mathbb{Q}$ and compact supports, $H^i(G, \mathbb{Q}(G/H))$. By Poincaré duality,

$$H_i(P, \mathbb{Q}) \cong H_c^{2-i}(P, \mathbb{Q}).$$

In particular, $H^2(G, \mathbb{Q}(G/H)) \cong H_0(P) = \mathbb{Q}$ since P is connected, $H^2(G, \mathbb{Q}(G/H)) \cong H_1(P) = \mathbb{Q}$ since $\pi_1(P) = H \cong \mathbb{Z}$ and $H^0(G, \mathbb{Q}(G/H)) \cong H_2(P) = 0$ since P is non-compact. □

We cannot use the same arguments for the conjugacy class $X_0 = 1$, since $Z(1) \neq \mathbb{Z}$. However, in this case we can simply calculate cohomology $H^i(G, \mathbb{Q}(X_0)) = H^i(G, \mathbb{Q})$ where G acts on $\mathbb{Q}$ trivially.

Since S has a contractible cover, it is an Eilenberg-MacLane space $S = K(G, 1)$, we know that $H^i(G, \mathbb{Q}) = H^i(S, \mathbb{Q})$. We can see that the element $h \in H^1(S)$ corresponds to the derivation

$$h : g \mapsto h(g)g,$$

where $h(g) \in \mathbb{Q}$ is the action of h on g viewed as a homology element.

We have shown that for any non-zero basis element $f_0 \neq f \in L$ which represents the conjugacy class X of G, the homomorphism $T \circ \rho|_{\mathbb{Q}f} : \mathbb{Q}f \to H^1(G, X)$ is an injective map of one-dimensional vector spaces and therefore an isomorphism.

Hence the map ρ_0 gives an isomorphism between L/γ_0 and the subspace of HH^1 corresponding to the direct sum

$$\bigoplus_{\{e\}\neq X\in[G]} H^1(G,X).$$

Note that this is precisely the set of "trace-zero" derivations, which map any element $g \in G \subset A$ to a linear combination of different elements of G. Since the trace of any commutator is zero, $\Im(\rho)$ is an ideal.

This lets us characterize the Goldman Lie algebra in terms of the algebraic structure of the Lie bracket on HH^1.

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