On Extremal Degrees of Minimal Ramsey Graphs

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July 26, 2011

Abstract

Let F, G and H be simple graphs. We say that F is (G, H)-Ramsey and write $F \to (G, H)$ if any coloring of the edges of F in red and blue contains either a red subgraph isomorphic to G or a blue subgraph isomorphic to H. Furthermore, if the above property is not retained after removing some edge or vertex from F, then F is called (G, H)-minimal. We define $s(G, H)$ to be the minimal degree of any vertex of a (G, H)-minimal graph, and $\bar{s}(G, H)$ to be the minimum possible maximum degree of any (G, H)-minimal graph. We prove that $\bar{s}(G, H) \geq \Delta(G) + \Delta(H) - 1 + \epsilon$, where $\Delta(G)$ is the maximum degree of any vertex of G and ϵ is the number of graphs in $\{G, H\}$ with even maximum degree. We also prove that $s(K_{1,m}, T) = 1$ for any positive integer m and tree T.

Summary

We consider simple graphs F with the property that however we color their edges in red and blue, they will contain either a red copy of G, or a blue copy of H, where G and H are known graphs. Furthermore, we require that after removing any edge from F, the above property no longer holds. Such graphs F we call (G, H)-minimal. We bound the minimum possible maximum degree of (G, H)-minimal graphs. We also generalize known values of the minimum degree of a vertex of any (G, G)-minimal graph to the asymmetric case.

1 Introduction

A folklore result states that any two-coloring of the edges of a complete graph on 6 vertices yields a monochromatic triangle. Furthermore, as shown in Figure 1.1, if any of the edges of the graph are removed, the above property no longer holds. More generally, if every two-coloring of the edges of a graph H contains a monochromatic copy of G, it is called *G-Ramsey*, after the famous mathematician. Furthermore, if no proper subgraph of H is G-Ramsey, then H is *G-minimal*.

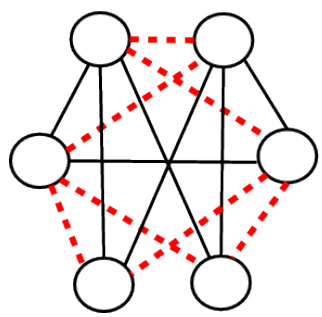

Figure 1.1: K_6 with an edge removed is not K_3-Ramsey

A 1930 result of Ramsey [1] states that for any graph G the set of G-Ramsey graphs, and thus the set of G-minimal graphs, is not empty:

Ramsey's Theorem. *For each pair of positive integers m and n, there exists an integer r such that any edge coloring of a complete graph on r vertices with the two colors red and blue contains either a red m-clique, or a blue n-clique as a subgraph.*

Considerable effort has gone into determining the minimum possible value of r for given m and n; this value is called *Ramsey number*. Very few Ramsey numbers are currently known, and the difference between the known upper and lower bounds for the rest is large [4]. The difficulty in calculating Ramsey numbers has brought up many related problems; a large portion of Ramsey theory deals with extremal properties of minimal graphs. These

properties include the Ramsey numbers $R(m, n)$, and the size Ramsey numbers. This paper will explore vertex degrees of minimal graphs.

1.1 Preliminaries

Unless otherwise stated, the graphs considered in this paper are simple.

Definition 1. *For any pair of graphs G_1 and G_2, we say that a graph H is* (G_1, G_2)-Ramsey*, or that H* arrows *G if every edge coloring of H in red and blue contains either a red copy of G_1, or a blue copy of G_2.*

Definition 2. *If a graph H is (G_1, G_2)-Ramsey and no proper subgraph H' of H is (G_1, G_2)-Ramsey, then H is called* (G_1, G_2)-minimal.

Definition 3. *The Ramsey number $R(G_1, G_2)$ is the minimum integer r such that the complete graph K_r is (G_1, G_2)-Ramsey.*

If H is (G_1, G_2)-Ramsey or (G_1, G_2)-minimal, we write $H \to (G_1, G_2)$ and $H \overset{m}{\to} (G_1, G_2)$ respectively. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$. Let $\mu(G_1, G_2) = \{H : H \overset{m}{\to} (G_1, G_2)\}$. For brevity, we write $H \to G$ and $H \overset{m}{\to} G$ for $H \to (G, G)$ and $H \overset{m}{\to} (G, G)$ respectively. K_a denotes a complete graph with a vertices, $K_{a,b}$ – a complete bipartite graph with a and b vertices in each partition, and C_a – a cycle with a vertices.

We will need two more definitions, which are the properties under consideration in this paper:

Definition 4.

$$s(G_1, G_2) = \min_{H \in \mu(G_1, G_2)} \delta(H) \tag{1}$$

Definition 5.

$$\bar{s}(G_1, G_2) = \min_{H \in \mu(G_1, G_2)} \Delta(H) \tag{2}$$

The following are multicolored generalizations of the above two definitions:

Definition 6. *We say that H is $((G_i)_{i=1..r})$-Ramsey if any coloring of the edges of H with one of the colors $\{c_1, c_2, ..., c_r\}$ yields a copy of G_i in color c_i for some i. If no proper subgraph of H has this property, then H is $((G_i)_{i=1..r})$-minimal.*

Definition 7. *We define the functions $s((G_i)_{i=1..r}) = \min \delta(H)$ and $\bar{s}((G_i)_{i=1..r}) = \min \Delta(H)$, where H can be any $((G_i)_{i=1..r})$-minimal graph.*

We also make the definitions $s(G) = s(G, G)$, $\bar{s}(G) = \bar{s}(G, G)$ and $\mu(G) = \mu(G, G)$.

1.2 Known results

The value of $s(G, H)$ was bounded by Fox and Lin [2]:

$$\delta(G) + \delta(H) - 1 \leq s(G, H) \leq R(G, H) - 1. \tag{3}$$

Trivially, the same bounds hold for $\bar{s}$ as well.

Burr et al. [5] proposed a proof that $s(K_a, K_b) = (a-1)(b-1)$. Fox and Lin [2] proved that for all positive integers a and b such that $a \leq b$, $s(K_{a,b}) = 2a - 1 = 2\delta(K_{a,b}) - 1$, and asked for what other graphs the lower bound (3) is tight. More recently, Szabó et al. [3] proved that for a larger class of bipartite graphs, including all bi-regular graphs, trees and even cycles, the lower bound $2\delta(G) - 1$ on the s-value is tight. They posed the question whether this lower bound is tight for all bipartite graphs.

Question (A)**.** *Is it true that for any bipartite graph G, $s(G) = 2\delta(G) - 1$?*

The smallest bipartite graph $BP*$ for which $s(BP*)$ is unknown is shown in Figure 1.2.

Yang [6] gave a necessary condition for any graph G for which $s(G) = 2\delta(G) - 1$. The property $\bar{s}(G)$ was studied by Burr et al. [5] who proved that $\bar{s}(K_a, K_b) = R(K_a, K_b) - 1$ for all pairs of complete graphs.

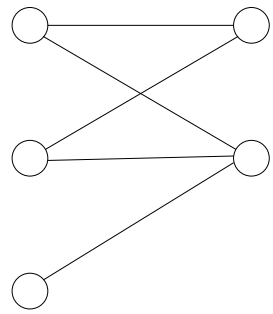

Figure 1.2: A graph $BP*$ for which $s(BP*)$ is unknown

For any graph G, the chromatic number $\chi(G)$ satisfies $\chi(G) \leq \Delta(G) - 1$. Burr et al. [5] proved that for all $n \geq 2$ the graph which arrows C_{2n+1} and has minimal chromatic number, has chromatic number no less than 5. Thus, it can be derived that $\bar{s}(C_{2n+1}) \geq 4$ if $n \geq 2$.

1.3 Problem definition and results

In section 2 we look into is the asymmetric variant of question A. We generalize Fox and Lin's result that $s(K_{a,b}) = 2\min(a,b) - 1$ to $s(K_{a,b}, K_{c,d}) = a + c - 1$ for any positive integers a, b, c, and d such that $a \leq b$ and $c \leq d$. Furthermore, we obtain that for any positive integer m and tree T, $s(K_{1,m}, T) = 1$.

We are also interested in the symmetric variant; however, instead of trying to answer it by finding the value $s(G)$ for different bipartite graphs G, we observe a relationship between the values of the functions s and $\bar{s}$. Suppose G is a graph, and G' is the same graph with one additional vertex, connected to one of the vertices in G. Intuitively, if a graph $H \to G$, and the degrees of the vertices of H are high enough, it is likely that $H \to G'$, and thus, if $\bar{s}(G)$ is high, attaching an edge would not change the value of the function s. On the other hand, if there exists a graph H with a small maximum degree which arrows G, the attached edge would most likely reduce the value of s. This motivates our study of the function $\bar{s}$. *In section 3* we obtain the lower bound $\bar{s}(G, H) \geq \Delta(G) + \Delta(H) - 3$. Furthermore, we improve the constant term 3 to 1 when $\Delta(G)$ and $\Delta(H)$ are odd, and to 2 if exactly one of them is

even. We also prove that if m and n are positive integers, $\bar{s}(K_{1,m}, K_{1,n}) = m + n - 1$ when m and n are odd, and $\bar{s}(K_{1,m}, K_{1,n}) = m + n - 2$ in case exactly one of the numbers m and n is odd.

2 On the value $s(G, H)$

Fox and Lin [2] note that their proof that $s(K_{a,b}) = 2a - 1$ if $a \leq b$ can be easily generalized for multiple colorings: $s(K_{a,b} : r) = ra - r + 1$ if $a \leq b$. We note that the same construction can be used for the asymmetric case:

Theorem 1. *For all pairs of positive integers $a_i \leq b_i$,*

$$s((K_{a_i,b_i})_{i=1..r}) = 1 - r + \sum_{i=1}^{r} a_i \tag{4}$$

Proof. Let $m = 1 - r + \sum_{i-1}^{r} a_i$. Then $K_{m,n} \to ((K_{a_i,b_i})_{i=1..r})$, provided n is large enough – the proof is the same as in Theorem 6 in [2], and uses the Pigeonhole principle to obtain that for some i at least a_i vertices in one of the partitions are connected to the same b vertices by monochromatic edges, where $b = \max_{i=1..r} b_i$. By construction, $\delta(K_{m,n}) = 1 - r + \sum_{i=1}^{r} a_i$. Since no subgraph of a complete bipartite graph can have a larger minimum degree, $s((K_{a_i,b_i})_{i=1}^{r}) \leq 1 - r + \sum_{i=1}^{r} a_i$. Equality holds due to the lower bound. □

Later in this section T is any tree. A generalization on the known fact that $s(T) = 1$ is:

Conjecture 1. *For any pair of trees T_1 and T_2, $s(T_1, T_2) = 1$.*

We prove this for the case when one of the trees is a star.

Theorem 2. *For any positive integer m and tree T, $s(K_{1,m}, T) = 1$.*

Proof. Let $\tau_{d,u}$ denote a complete d-ary rooted tree with depth u. Then for a large enough u, the tree $\tau_{\Delta(T),u}$ contains a copy of T. Consider any red-blue coloring of the edges of

$\tau_{m+\Delta(T)-1,u}$. Suppose it does not contain a red copy of $K_{1,m}$. Then each vertex would have at least $\Delta(T)$ blue children, necessitating a blue copy of $\tau_{\Delta(T),u}$. Thus, $\tau_{m+\Delta(T)-1,u} \to (K_{1,m}, \tau_{\Delta(T),u})$. This implies that $\tau_{m+\Delta(T)-1,u} \to (K_{1,m}, T)$. Any subgraph of a tree has a vertex of degree 1, so there exists a $(K_{1,m}, T)$-minimal graph with a vertex of degree 1. □

3 On the value of $\bar{s}(G)$ and $\bar{s}(G, H)$

We prove a lower bound for $\bar{s}(G)$, analogous to the lower bound for $s(G)$ (3):

Theorem 3. *For any graph G, $\bar{s}(G) \geq 2\Delta(G) - 2$. Furthermore, if $\Delta(G)$ is odd, $\bar{s}(G) \geq 2\Delta(G) - 1$.*

Proof. Case 1: $\Delta(G)$ is odd. Suppose there exists a graph $H \to G$ and $\Delta(H) \leq 2\Delta(G) - 2$. If H is not connected, we process each component separately. Thus, we may assume that H is connected. Let H' be obtained from H by adding edges so that each vertex is of degree $2\Delta(G) - 2$. To ensure that this can always be done, we do not require H' to be simple. Then H' contains an Euler cycle of even length, because every vertex of H has degree $2\Delta(G) - 2$, which is divisible by 4. If we color the Euler cycle with alternating blue and red edges, the number of red and blue edges incident upon each vertex of H' is the same. If we color each edge of H the same color as the corresponding edge of H', then each vertex of H would have at most $\Delta(G) - 1$ incident edges of the same color. Thus, it does not contain a monochromatic copy of G, contradicting out assumption that $H \to G$.

Case 2: $\Delta(G)$ is even. Suppose there exists a graph H, such that $H \to G$ and $\Delta(H) \leq 2\Delta(G) - 3$. We construct a $(2\Delta(G) - 2)$-regular graph H' containing a copy of H as explained in case 1. There exists an Euler cycle in H'. If it has even length, the proof is identical to the proof for case 1. If the Euler cycle has odd length, we remove one of its edges which is not present in H, which exists by construction, and proceed with the proof as in case 1. □

Although the above proof uses symmetry, we make the intuitive conjecture for the asymmetric case:

Conjecture 2.

$$\bar{s}(G,H) \geq \begin{cases} \Delta(G)+\Delta(H)-1, & \textit{if } \Delta(G) \textit{ and } \Delta(H) \textit{ are odd,} \\ \Delta(G)+\Delta(H)-2, & \textit{otherwise.} \end{cases}$$

We prove a weaker version of this conjecture:

Theorem 4.

$$\bar{s}(G,H) \geq \begin{cases} \Delta(G)+\Delta(H)-1, & \textit{if } \Delta(G) \textit{ and } \Delta(H) \textit{ are odd,} \\ \Delta(G)+\Delta(H)-2, & \textit{if exactly one of } \Delta(G) \textit{ and } \Delta(H) \textit{ is even,} \\ \Delta(G)+\Delta(H)-3, & \textit{if } \Delta(G) \textit{ and } \Delta(H) \textit{ are even.} \end{cases}$$

Proof. Case 1: $\Delta(G)$ *and* $\Delta(H)$ *are odd.* Let $r = \Delta(G)+\Delta(H)-2$. Suppose there exists a graph $F \to (G,H)$ such that $\Delta(F) \leq r$. We shall construct an r-regular graph F' containing a copy of F. Let R_r denote the graph obtained from K_{r+1} by removing an edge, as shown in Figure 3.1. Later in this proof we call the two vertices which were incident upon the removed edge *ends*. In Figure 3.1 the ends are denoted with s and t. The degree of each vertex of R_r except for the ends is r. We will repeatedly modify F using the following algorithm.

1. If there is a vertex $v \in V(F)$ such that $deg(v) \leq r-2$, we add a copy of R_r and connect its ends to v.

2. If $\delta(F) = r-1$, let $v_1 \in V(F)$ be a vertex with degree $r-1$. Because the sum of the degrees in any graph is even, there is a vertex $v_2 \in V(F)$ such, that $v_2 \neq v_1$ and

$deg(v_2) = r - 1$. Then we add a copy of R_r and connect one of its ends to v_1, and the other to v_2.

3. Otherwise, every vertex is of degree r, and we have obtained F'.

The above algorithm will halt, because the nonnegative value $\sum_{v \in H}(\deg(v) - 6)$ is decreasing after each step. Since r is even, F' is 2-factorable [7]. Then we can color $(\Delta(G) - 1)/2$ of the 2-factors red, and $(\Delta(F) - 1)/2$ factors blue. The resulting coloring does not contain a red copy of G, nor a blue copy of H, because each vertex has at most $\Delta(G) - 1$ red and $\Delta(H) - 1$ blue edges. Since a copy of F is contained in F', $F \not\to (G, H)$.

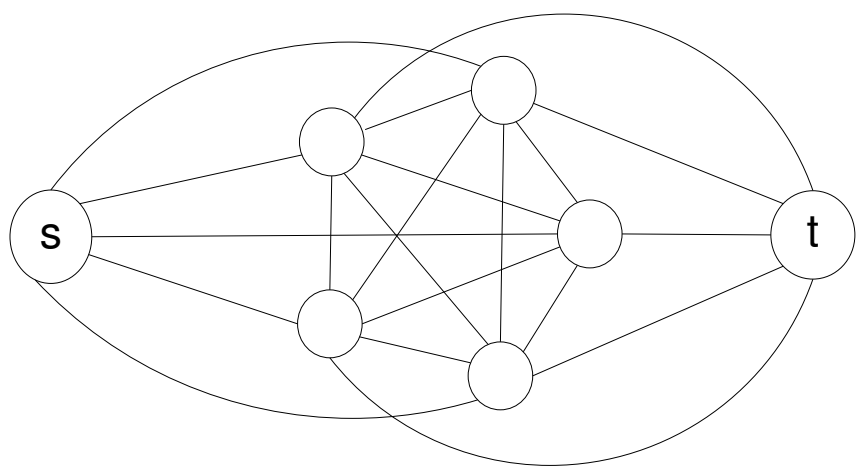

Figure 3.1: R_6

Case 2: $\Delta(G)$ is odd and $\Delta(H)$ is even. To prove this case we are going to use the following properties of Ramsey graphs:

$$\bar{s}(G, H) \geq \bar{s}(K_{1,\Delta(G)}, K_{1,\Delta(H)}) \tag{5}$$

$$\bar{s}(K_{1,m}, K_{1,n}) \leq \bar{s}(K_{1,m}, K_{1,n+1}) \tag{6}$$

In the next equation, the left-hand inequality follows from (5) and (6), while the right-hand

inequality – from case 1 of this theorem.

$$\bar{s}(G,H) \geq \bar{s}(K_{1,\Delta(G)}, K_{1,\Delta(H)-1}) \geq \Delta(G) + \Delta(H) - 2 \tag{7}$$

Case 3: $\Delta(G)$ and $\Delta(H)$ are even. The proof for this case is analogous to the proof of case 2. □

We now prove two corollaries, related to this Theorem.

Corollary 1 (multicolored generalization)**.** *If c is the number of colors,*

$$\bar{s}((G_i)_{i=1..c}) \geq 1 - \epsilon + \sum_{i=1}^{c} (\Delta(G_i) - 1) \tag{8}$$

where

$$\epsilon = |G_i : \Delta(G_i) \text{ is even}|$$

Proof. Case 1: $\Delta(G_i)$ is odd for all i. Let $r = \sum_{i=1}^{c}(\Delta(G_i)-1)$. If $r \leq 0$, the inequality holds. Otherwise, let F be a graph such that $\Delta(F) \leq r$ and $F \to ((G_i)_{i=1..c})$. We build an r-regular graph F', containing a copy of F, as explained in the proof of Theorem 4. We 2-factor F' and color $(\Delta(G_i) - 1)/2$ factors in the i-th color for every i. The resulting coloring does not contain a copy of G_i in the i-th color.

Case 2: $\Delta(G_i)$ is even if and only if $i \leq \epsilon$ for some $\epsilon \leq c$. Then,

$$\bar{s}((G_i)_{i=1..r}) \geq \bar{s}(K_{1,\Delta(G_1)-1}, K_{1,\Delta(G_2)-1}, ..., K_{1,\Delta(G_\epsilon)-1}, K_{1,\Delta(G_{\epsilon+1})}, K_{1,\Delta(G_{\epsilon+2})}, ..., K_{1,\Delta(G_c)}) \geq 1 - \epsilon + \sum_{i=1}^{c}(\Delta(G_i) - 1)$$

□

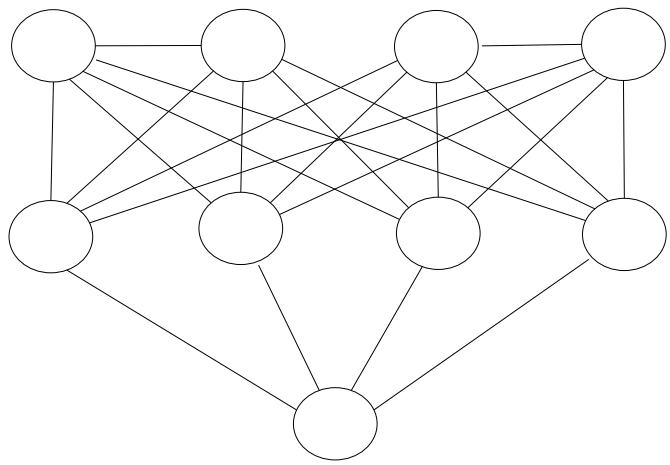

Figure 3.2: ρ_5

Corollary 2. *For any two positive integers m and n the following holds:*

$$\bar{s}(K_{1,m}, K_{1,n}) \begin{cases} = m+n-1, \text{ if } m \text{ and } n \text{ are odd} & \text{(case 1)} \\ = m+n-2, \text{ if exactly one of } m \text{ and } n \text{ is odd} & \text{(case 2)} \\ \leq m+n-2, \text{ if both } m \text{ and } n \text{ are even} & \text{(case 3)} \end{cases}$$

Proof. Case 1: Equality follows from Theorem 4 and the fact that $K_{1,m+n-1} \to (K_{1,m}, K_{1,n})$.

Case 2: Let $r = m+n-2$, which is odd. We construct an r-regular graph which arrows $(K_{1,m}, K_{1,n})$. Let the graph ρ_r be constructed from the complete bipartite graph $K_{r-1,r-1}$ by adding one vertex and connecting it to all the vertices in one of the partitions, and then splitting the vertices in the other partition in pairs and connecting each pair with an edge, as shown on Figure 3.2. We call the added vertex an *end*.

Consider the graph F consisting of r distinct copies of ρ_r and one additional vertex, connected to their ends. We enumerate the ends with the integers 1 through n. Let r_i be the copy of ρ_r with end i. We call the vertex not belonging to any of these copies of ρ_r the *central vertex*. Suppose that $F \not\to (K_{1,m}, K_{1,n})$. Then it can be colored so that each vertex has exactly $m-1$ red and $n-1$ blue incident edges. Let v be a vertex connected to the

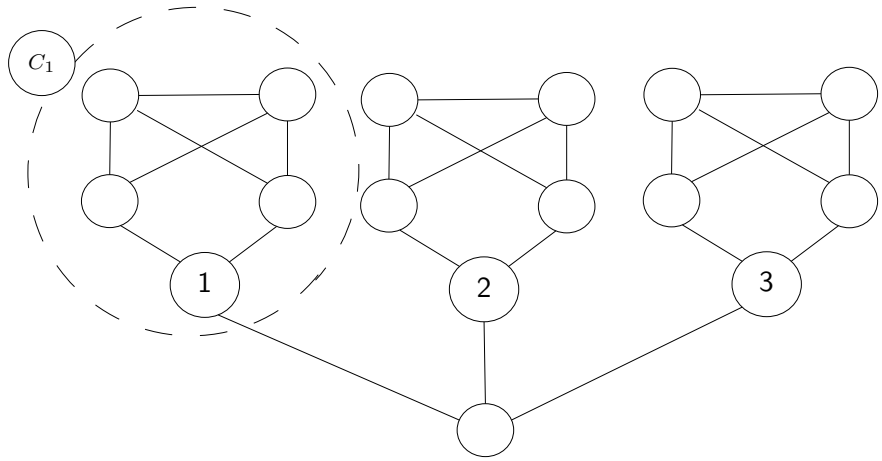

Figure 3.3: A graph which is $(K_{1,2}, K_{1,3}) - Ramsey$

central vertex by a blue edge. Then the subgraph r_v has an odd number of vertices and each of them has an odd number of incident red edges, a contradiction, since the sum of the degrees of the vertices of any graph is even.

Case 3: $K_{m+n-1} \to (K_{1,m}, K_{1,n})$, because it has odd number of vertices, so you can't color the edges so that each vertex is incident to $m-1$ red and $n-1$ blue edges, as $m-1$ and $n-1$ are odd. Thus, $\bar{s}(K_{1,m}, K_{1,n}) \leq m+n-1$. $\square$

4 Conclusion

We generalized known values of the function $s(G)$ to the asymmetric case. We also found a lower bound on the value $\bar{s}(G, H)$ and the specific value $\bar{s}(K_{1,m}, K_{1,n})$ for some stars. Future study may work on Conjectures 1 and 2 or improve the current known upper bound on $s(G)$, which is $R(G)$.

If Conjecture 2 is true, then $\bar{s}(K_{1,m}, K_{1,n}) = m+n-2$ for even m and n, classifying $\bar{s}(K_{1,m}, K_{1,n})$ for all pairs of stars. Furthermore, it may lead to an improvement of Corollary 1.

5 Acknowledgments

Thanks to Wuttisak Trongsiriwat, my mentor, who formulated the project and made many useful suggestions, Dr. David Wildstrom, my tutor, Dr. Tanya Khovanova, Dr. Evgeniya Sendova, and the rest of the RSI staff, for supervising and revising my paper. My deepest gratitute to Dr. Jerison, Kartik Venkatram, MIT, and MIT Math Department, for organizing the math mentorships at RSI 2011. Also, had it not been for the Center for Excellence in Education and the Research Science Institute, I would not have been able to do this research. I would as well like to thank America for Bulgaria Foundation and Sts. Cyril and Metodius International Foundation for sponsoring my stay at RSI, as well as all RSI sponsors. Last, but not least, thanks to the High School Science Institute in Bulgaria for organizing research opportunities in my country and my trip to RSI.

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