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18.117 PS 3
Due 4/08/05

1. Let X be a compact n-dimensional manifold equipped with a smooth non-vanishing measure dx. Let V be the volume of X. Suppose $K \in C^\infty(X \times X)$ and there exists $0 < \varepsilon < 1$ such that $|K(x,y)| < \varepsilon/V$ for all x and y. Let $K^i = K \circ \cdots \circ K$ (i factors) where $K_1 \circ K_2$ is defined by

$$K_1 \circ K_2(x,y) = \int_X K_1(x,z)K_2(z,y)\,dz$$

so that $T_{K_1}T_{K_2} = T_{K_1K_2}$. Note that if $K_1 < c_1$ and $K_2 < c_2$ on $X \times X$ then $K_1 \circ K_2 < Vc_1c_2$ on $X \times X$ because the integrand is bounded by the constant c_1c_2. Thus

$$K^1 = K < \varepsilon/V, \quad K^2 = K \circ K < V(\varepsilon/V)^2 = \varepsilon^2/V, \quad \ldots, \quad K^i < \varepsilon^i/V$$

on $X \times X$. Thus if we set $L(x,y) = -\sum_{i=1}^\infty K^i(x,y)$, this sum converges uniformly on $X \times X$. Moreover, note that

$$\frac{\partial}{\partial x_i}(K_1 \circ K_2(x,y)) = \int_X \frac{\partial}{\partial x_i}K_1(x,z)K_2(z,y)\,dz = \Big(\frac{\partial}{\partial x_i}K_1\Big) \circ K_2(x,y),$$
$$\frac{\partial}{\partial y_i}(K_1 \circ K_2(x,y)) = \int_X \frac{\partial}{\partial y_i}K_1(x,z)K_2(z,y)\,dz = K_1 \circ \Big(\frac{\partial}{\partial y_i}K_2\Big)(x,y),$$

so for $i \geq 2$,

$$D_x^\alpha D_y^\beta K^i(x,y) = K_x^\alpha \circ K^{i-2} \circ K_y^\beta(x,y)$$

where $K_x^\alpha(x,y) = D_x^\alpha K(x,y)$, $K_y^\beta(x,y) = D_y^\beta K(x,y)$. Now since K_x^α and K_y^β are continuous on $X \times X$, hence bounded, and $K^{i-2} < \varepsilon^{i-2}/V$, so the sum $\sum_{i=1}^\infty D_x^\alpha D_y^\beta K(x,y)$ also converges uniformly on $X \times X$ for all α and β; thus $L(x,y) \in C^\infty(X \times X)$. Finally,

$$(I - T_K)(I - T_L) = (I - T_K)(I + T_K + T_K^2 + T_K^3 + \cdots) = \lim_{n\to\infty} I - T_K^n = I$$

and similarly $(I - T_L)(I - T_K) = I$. Hence $I - T_K$ is invertible with inverse of the form $I - T_L$ with $L \in C^\infty(X \times X)$.

2. (a) Let T_K be a finite rank smoothing operator and let T_L be any smoothing operator. Write $K(x,y) = \sum_{i=1}^N f_i(x)g_i(y)$; then

$$K \circ L = \int_X \sum_{i=1}^N f_i(x)g_i(z)L(z,y)\,dz = \sum_{i=1}^N f_i(x)\Big(\int_X g_i(z)L(z,y)\,dz\Big)$$

and

$$L \circ K = \int_X L(x,z)\sum_{i=1}^N f_i(z)g_i(y)\,dz = \sum_{i=1}^N \Big(\int_X L(x,z)f_i(z)\,dz\Big)g_i(y)$$

so T_KT_L and T_LT_K are finite rank smoothing operators.

(b) First suppose $f \in \ker I - T_K$. Then $f(x) = T_Kf(x)$ so

$$f(x) = \int_X \sum_{i=1}^n f_i(x)g_i(y)f(y)\,dy = \sum_{i=1}^n f_i(x)\Big(\int_X g_i(y)f(g)\,dy\Big)$$

hence f is in the linear span of the f_i, so $\ker I - T_K$ is finite dimensional. Now suppose $\int_X f(y)g_i(y)\,dy = 0$ for $i = 1, \ldots, N$; then

$$(I - T_K)f(x) = f(x) - \int_X \sum_{i=1}^N f_i(x)g_i(y)f(y)\,dy = f(x) - \sum_{i=1}^N f_i(x)\Big(\int_X g_i(y)f(y)\,dy\Big) = f(x)$$

so f is in the image of $I - T_K$. Thus the image of $I - T_K$ contains the finite-codimensional subspace defined by the equations $\int_X f(y)g_i(y)\,dy$, hence the cokernel of $I - T_K$ is finite dimensional.

3. Let $\mathcal{A} \subset C(X \times X)$ be the set of all functions of the form

$$K(x,y) = \sum_{i=1}^{N} f_i(x) g_i(y)$$

where f_i and g_i are C^∞ functions on X. Clearly $\mathcal{A}$ is closed under addition. Also $\mathcal{A}$ is closed under multiplication, because

$$\Big(\sum_{i=1}^{N} f_i(x) g_i(y)\Big)\Big(\sum_{j=1}^{N'} f'_j(x) g'_j(y)\Big) = \sum_{i=1}^{N}\sum_{j=1}^{N'} (f_i f'_j)(x)(g_i g'_j)(y).$$

Finally, $\mathcal{A}$ is clearly real and constant functions are evidently in $\mathcal{A}$. If $(x_1, y_1) \neq (x_2, y_2)$ are two distinct points of $X \times X$, then either $x_1 \neq x_2$ or $y_1 \neq y_2$; assume without loss of generality that $x_1 \neq x_2$. We can choose a function $f \in C^\infty(X)$ such that $f(x_1) = 1$ and $f(x_2) = 0$; taking $g(y) = 1$, then $f(x_1)g(y_1) = 1$ and $f(x_2)g(y_2) = 0$, so $f(x)g(y)$ separates (x_1, y_1) and (x_2, y_2), and it is in $\mathcal{A}$. Thus $\mathcal{A}$ satisfies the conditions of the Stone-Weierstrass theorem, so $\mathcal{A}$ is dense in $C(X \times X)$. In particular if $K \in C^\infty(X \times X)$ and $\varepsilon > 0$ then there exists a function $K_1 \in \mathcal{A}$ such that $\sup |K - K_1|(x,y) < \varepsilon$.

4. Let T_K be a smoothing operator. Choose $0 < \varepsilon < 1$; by exercise 3 there exists $K_1 \in C^\infty(X \times X)$ such that T_{K_1} is of finite rank and $|K - K_1| < \varepsilon / V$, where V is the volume of X. Set $K_2 = K - K_1$; then K_2 satisfies the hypotheses of exercise 1, so $I - T_{K_2}$ has an inverse $I - T_L$. Then $I - T_K = (I - T_{K_2}) - T_{K_1}$ so by exercise 2,

$$(I - T_K) \circ (I - T_L) = ((I - T_{K_2}) - T_{K_1}) \circ (I - T_L) = I - T_{K_1} \circ (I - T_L)$$

is of the form identity minus a finite rank smoothing operator, hence has finite dimensional cokernel. The image of $I - T_K$ contains the image of $(I - T_K) \circ (I - T_L)$, so $I - T_K$ also has finite dimensional cokernel. Similarly

$$(I - T_L) \circ (I - T_K) = (I - T_L) \circ ((I - T_{K_2}) - T_{K_1}) = I - (I - T_L) \circ T_{K_1}$$

is of the same form, hence has finite dimensional kernel, and since the kernel of $I - T_K$ is contained in the kernel of $(I - T_L) \circ (I - T_K)$, $I - T_K$ has finite dimensional kernel as well. Therefore $I - T_K$ has finite dimensional kernel and cokernel.