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Lecture 1:
Introduction; Couloumb's law; Superposition; Electric energy

1.1 Fundamental forces

The electromagnetic force is one of the four fundamental forces of physics. Let's compare it to the others to see where it fits in the grand scheme of things:

Force	Range (cm)	Interaction particles	Exchange particles	"Strength"
Gravity	∞	All mass/energy	graviton ($m = 0$)	1
Weak	10^{-15}	All elementary particles except photon & gluons	weak bosons ($m \sim 100 \times m_{\rm proton}$)	10^{24}
Electromagnetic	∞	All charges	photon ($m = 0$)	10^{35}
Strong	10^{-13}	quarks and gluons	gluons ($m = 0$)	10^{37}

[The "strength" entries in this table reflect the relative magnitudes of the various forces as they act on a pair of protons in an atomic nucleus. Note that MIT professor Frank Wilczek shared the 2004 physics Nobel Prize with David Gross (now director of the Kavli Institute for Theoretical Physics at UC Santa Barbara) and H. David Politzer (now professor of physics at Caltech) for work that made possible our current understanding of the strong force.]

Let's summarize the major characteristics of the electromagnetic force:

- **Strong**: It is the 2nd strongest of the fundamental forces — only the appropriately named strong force (which binds quarks and nuclei together) beats it (and only at very small distances).
- **Long Range**: As we'll discuss soon, the electromagnetic force has the form $F \propto 1/r^2$, just like gravity. This means it has *infinite range*: it gets weaker with distance, but just slowly enough that things really far away still feel it. Contrast this with the weak force, which has the form $F \propto \exp(-r/R_{\rm W})/r^2$, where $R_{\rm W} \simeq 10^{-15}$ cm — it dies away REALLY fast with r.
- **Has 2 signs**: It can attract and repel. Contrast this with gravity which only attracts.
- **Acts on charge**: More on this shortly.

The electromagnetic interaction plays an *extremely* important role in everyday life. The forces that govern interactions between molecules in chemistry and biology (e.g., ionic forces, van der Waals, hydrogen bonding) are all fundamentally electromagnetic in nature. A thorough understanding of electromagnetism will carry you far in understanding the properties of things you encounter in the everyday world. Add quantum mechanics and you've got most of the tools needed to completely understand these things.

1.2 History of electromagnetism

Circa B.C. 500: Greeks discover that rubbed amber attracts small pieces of stuff. (Note the Greek word for amber: "$\eta\lambda\epsilon\kappa\tau\rho o\nu$", or "electron".) They also discover that certain iron rich rocks from the region of $\mathrm{M}\alpha\gamma\nu\eta\sigma\iota\alpha$ (Magnesia) attract other pieces of iron.

1730: Charles Francois du Fay noted that electrification seemed to come in two "flavors": *vitreous* (now known as positive; example, rubbing glass with silk) and *resinous* (negative; rubbing resin with fur)

1740: Ben Franklin suggested the "one fluid" hypothesis: "positive" things have more charge than "negative" things. Suggested an experiment to Priestley to *indirectly* measure the inverse square law...

1766: Joseph Priestley: did the suggested experiment, indirectly proving $1/r^2$ form of the force law.

1773: Henry Cavendish: did Priestley's experiment accurately:

$$F = k\frac{q_1 q_1}{r^{2+\delta}}\ , \qquad |\delta| < \frac{1}{50}\ .$$

(By the late 1800s, Maxwell was able to show that $|\delta| < 1/21600$; by 1936 Plimpton and Lawton were able to show $|\delta| < 2 \times 10^{-9}$.)

1786: Charles-Augustin de Coulomb measured the electrostatic force and *directly* verified the inverse square law.

1800: Count Alessandro Giuseppe Antonio Anastasio Volta invented the electric battery.

1820: Hans Chrstian Oersted and André-Marie Ampère establish connection between magnetic fields and electric currents

1831: Michael Faraday discovers magnetic induction

1873: James Clerk Maxwell unified electricity and magnetism into electromagnetism.

1887: Heinrich Hertz confirms the connection between electromagnetism and radiation.

1905: Albert Einstein formulates the special theory of relativity, which (among other things) clarifies the inter-relationship between electric and magnetic fields.

1.3 Electric charge

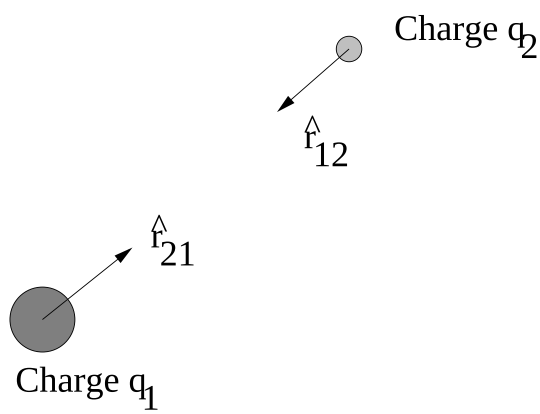
As stated above, the electromagnetic interaction acts on *charges*. This is really a tautological statement: all we are doing is defining "charge" as the property that the electromagnetic interaction interacts with. We accept this and move on.

In nature, we find that electric charge comes in discrete lumps or "quanta". This was first measured by Robert Millikan in 1923 in his famous oil drop experiment. The smallest charge we ever observe is the "elementary charge" e; it takes the value 1.602×10^{-19} Coulombs (one of the units of charge we will use), or 4.803×10^{-10} esu (electrostatic units). The magnitude of the electron's charge and of the proton's charge is e. The charges we encounter in nature come in opposite signs. By convention, the electron has negative sign, the proton positive.

Charge is conserved: in any isolated system, the total charge cannot change. If it does change, then the system is not isolated: charge either went somewhere or came in from somewhere.

1.4 Coulomb's law

Coulomb's law tells us that the force between two charges is proportional to the product of the two charges, inversely proportional to the distance between them, and points along the vector pointing from one charge to the other:



The unit vector $\hat{r}_{21}$ points *toward* 2 *from* 1; $\hat{r}_{12}$ points *toward* 1 *from* 2. This means that $\hat{r}_{12} = -\hat{r}_{21}$. We write the distance to charge 2 from 1 as r_{21}; this must be exactly the same as the distance to charge 1 from 2, r_{12}.

According to Coulomb's law, the force that charge q_2 feels due to charge q_1 is given by

$$\vec{F}_2 = k\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}$$

(We'll discuss the constant k shortly. The key thing to note right now is that it is *positive*.) Likewise, the force q_1 feels due to charge q_2 is

$$\begin{aligned}\vec{F}_1 &= k\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}\\ &= -k\frac{q_1 q_2}{r_{12}^2}\hat{r}_{21}\\ &= -\vec{F}_2\ .\end{aligned}$$

The force on q_1 is equal but opposite to the force on q_2. Newton would be pleased (3rd law).

It's worth noting that we are *assuming* Coulomb's law. Of course, this assumption is justified because many many experiments have verified that this is the way the electric force works. It's kind of interesting, though, that pretty much this entire class follows from this assumption. [Indeed, there are only three other "fundamental" assumptions that we will need — superposition (discussed momentarily), special relativity, and the non-existence of magnetic charge. All of 8.022 can be derived from these starting points!]

To illustrate Coulomb's law, we have introduced all these subscripts and made it seem like half of our time is going to be spent labeling charges, unit vectors, and distances. In practice, though, the thing to note is that the force *in both cases* is along the vector from one charge to the other. Its direction in both cases is *repulsive* if both charges have the same sign; it is *attractive* if the charges are of opposite sign. Like charges repel; opposites attract.

1.5 Units

In electrodynamics, the choice of units we use matters: the basic equations actually have slightly different forms in different system of units. Contrast this with mechanics, where $\vec{F} = m\vec{a}$ regardless of whether we measure length in units of meters, inches, furlongs, or lightyears.

The system of units that is used by our textbook, and that is common in much physics research (particularly theory) is the *CGS* unit system: length is measured in centimeters, masses in grams, time in seconds. In this system of units the constant k appearing in Coulomb's law is equal to 1. The cgs unit of charge is the *esu*, an acronym for the oh-so-clever name *e*lectro*s*tatic *u*nit. It is defined such that two charges of 1 esu each, separated by 1 cm, feel a force of 1 dyne:

$$1\,\text{dyne} = \frac{(1\,\text{esu})^2}{(1\,\text{cm})^2} \to 1\,\text{esu} = \text{cm}\sqrt{\text{dyne}}\,.$$

Another system of units that is commonly encountered is known as *SI* (for Systèm Internationale) or *MKS* units. Engineers typically use SI units in their work. Lengths are measured in meters, masses in kilograms, time in seconds. Charges are measured in *Coulombs*, abbreviated C. In this system of units the constant k in Coulomb's law takes the value

$$k = \frac{1}{4\pi\epsilon_0}$$

where ϵ_0 is yet another constant called the permittivity of free space. It takes the value

$$\epsilon_0 = 8.854 \times 10^{-12}\,\text{C}^2\,\text{N}^{-1}\,\text{m}^{-2}\,.$$

The constant k then takes the value

$$\begin{aligned} k &= 8.988 \times 10^9\,N\,\text{C}^{-2}\text{m}^2 \\ &\simeq 9 \times 10^9\,N\,\text{C}^{-2}\text{m}^2 \end{aligned}$$

Note that the force between two 1 Coulumb charges held a meter apart is 9 *billion* Newtons! 1 Coulomb is an enormous amount of charge: 1 Coulomb = 2.998×10^9 esu. (The 2.998 is often approximated 3. Note that the speed of light's value is 2.998×10^{10} cm/sec. The equivalence of the numerical factors is *not* a coincidence!)

A common complaint in this class is that "no one uses cgs units, so why should we"? This complaint is not really valid — people do in fact use cgs for a *lot* of things, as well as even more bizarre variants of cgs units. (For example, in my research I work in a version of cgs units in which both time and mass are measured in centimeters. In this system, the speed of light c and the gravitational constant G are both equal to exactly 1. In some cases, people actually *mix* units. The most commonly used unit for magnetic field is the Gauss, which is cgs — even if everything else is measured in SI. Never count on people to adopt a logical system of anything!)

It is true, however, that cgs units are not encountered as often as SI units, so I understand this annoyance. The harsh truth is that the best thing to do is to adapt, learn how to convert between different unit conventions, and deal with it. It's actually good practice for being a scientist or engineer — you'd be stunned how often people use what might be considered nonstandard choices for things, especially if you have to dig back into older documents.

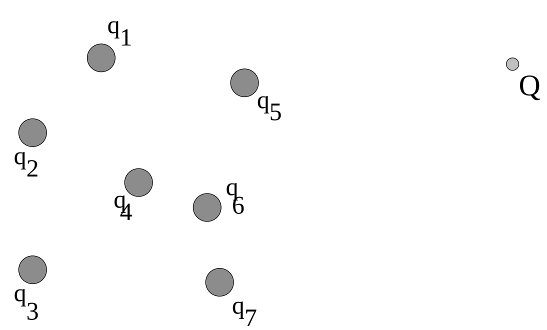
Here is a summary of how to convert various quantities between SI and CGS. In this table, "3" is shorthand for 2.998. (More precisely, 2.99792458). "9" means "3" squared, or 8.988. (More precisely, 8.9875517873681764.) In the vast majority of calculations, approximating "3" as 3 and "9" as 9 is plenty accurate.

	SI Units		CGS units
Energy	1 Joule	=	10^7 erg
Force	1 Newton	=	10^5 dyne
Charge	1 Coulomb	=	"3" $\times 10^9$ esu
Current	1 Ampere	=	"3" $\times 10^9$ esu/sec
Potential	"3" $\times 10^2$ Volts	=	1 statvolt
Electric field	"3" $\times 10^4$ Volts/m	=	1 statvolt/cm
Magnetic field	1 Tesla	=	10^4 gauss
Capacitance	1 Farad	=	"9" $\times 10^{11}$ cm
Resistance	"9" $\times 10^{11}$ Ohm	=	1 sec/cm
Inductance	"9" $\times 10^{11}$ Henry	=	1 sec^2/cm

1.6 Superposition

The principle of superposition is a fancy name for a simple concept: in electromagnetism, forces add.

More concretely: suppose we have a number N of charges scattered in some region. We want to calculate the force that *all* of these charges exert on some "test charge" Q. Label

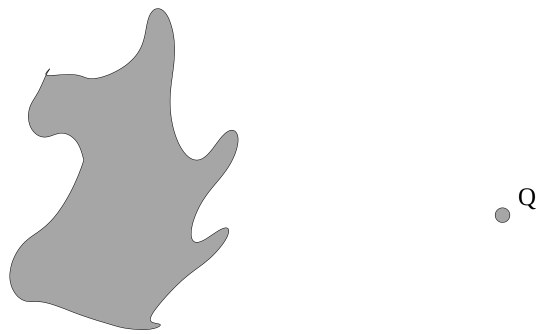


the ith charge q_i; let r_i be the distance from q_i to Q. Let $\hat{r}_i$ be the unit vector pointing from q_i to Q. The total force acting on Q is just the total found by summing the forces from each *individual* charge q_i:

$$\begin{aligned} \vec{F}_Q &= \frac{Qq_1\hat{r}_1}{r_1^2} + \frac{Qq_2\hat{r}_2}{r_2^2} + \frac{Qq_3\hat{r}_3}{r_3^2} + \ldots \\ &= \sum_{i=1}^{N} \frac{Qq_i\hat{r}_i}{r_i^2} \, . \end{aligned}$$

Since we know calculus, we can take a limiting description of this calculation. Let's imagine that, in our swarm of charges, the number N becomes infinitely large, the charges all press together, and the whole thing goes over to a *continuum*. We describe this continuum by the *charge density* or charge per unit volume ρ.

Imagine that we've got a big blob of charged goo with density ρ sitting somewhere. We bring in our test charge Q close to it. How do we calculate the total force acting on the test charge?



Simple: we chop the blob up into little chunks of volume ΔV; each such chunk contains charge $\Delta q = \rho \Delta V$. Suppose there are N total such chunks, and we label each one with some index i. Let $\hat{r}_i$ be the unit vector pointing from the ith chunk to the test charge; let r_i be the distance between chunk and test charge. The total force acting on the test charge is

$$\vec{F} = \sum_{i=1}^{N} \frac{Q(\rho \Delta V_i)\hat{r}_i}{r_i^2}$$

This formula is really an approximation, since we probably can't perfectly describe our blob by a finite number of little chunks. The approximation becomes *exact* if we let the number of chunks go to infinity and the volume of each chunk go to zero — the sum then becomes an integral:

$$\vec{F} = \int_V \frac{Q\,\rho\,dV\,\hat{r}}{r^2}\ .$$

If the charge is uniformly smeared over a surface, then we integrate a *surface* charge density σ over the area of the surface:

$$\vec{F} = \int_A \frac{Q\,\sigma\,dA\,\hat{r}}{r^2}\ .$$

If the charge is uniformly smeared over a line, then we integrate a *line* charge density λ over the length:

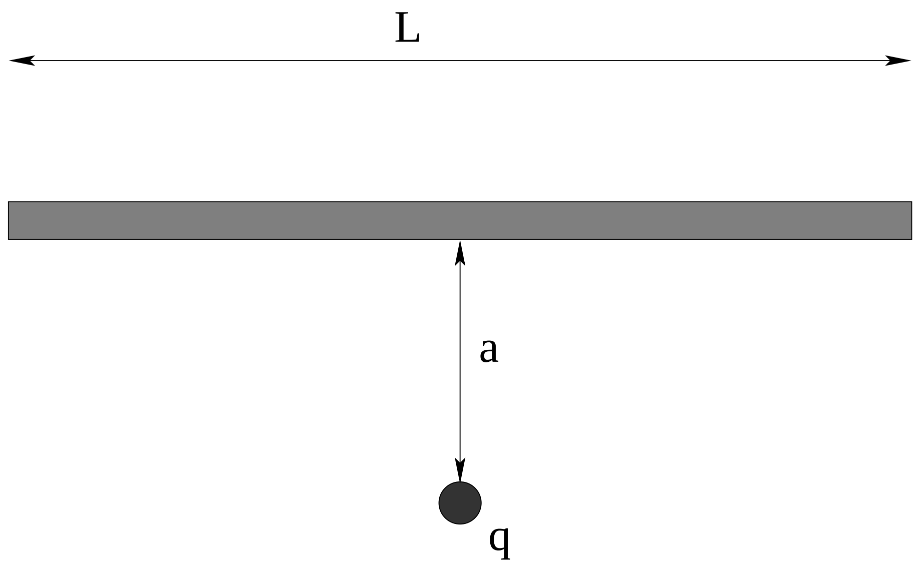
$$\vec{F} = \int_L \frac{Q\,\lambda\,dl\,\hat{r}}{r^2}\ .$$

(Note that the letter σ is the Greek equivalent for s, so it makes sense for a surface distribution. λ is the Greek equivalent for l, so it makes sense for a line distribution. ρ is the Greek equivalent for r so it makes sense ... actually, it makes no sense at all. It was almost logical.)

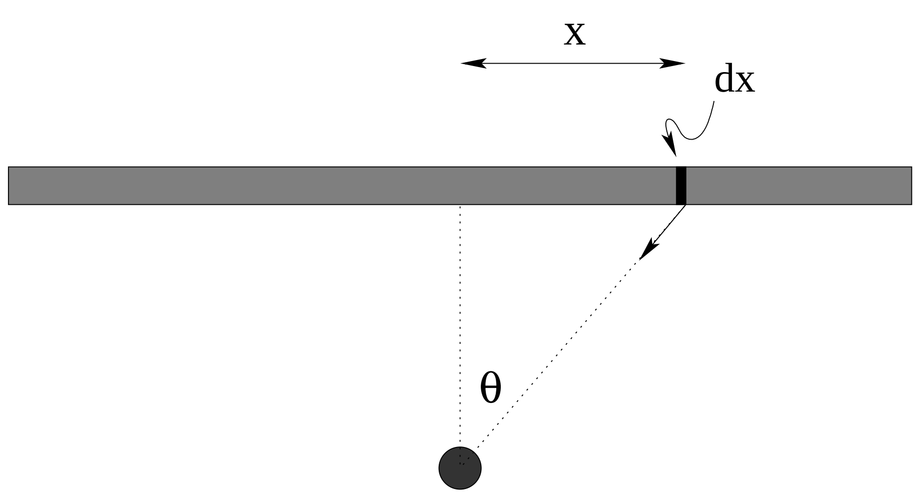
A good portion of the calculations that we will do will be like this. Let's look at an example:

1.6.1 Example: Force on a test charge q due to a uniform line charge

A rod of length L has a total charge Q smeared uniformly over it. A test charge q sits a distance a away from the rod's midpoint. (Ignore the thickness of the rod.) What is the electric force that the rod exerts on the test charge?



First, define a coordinate system: let the direction parallel to the rod be x, let the perpendicular direction be y. Next, imagine dividing the rod up into little pieces of width dx:



In drawing things this way, we have defined $x = 0$ as the rod's midpoint, so the rod runs from $x = -L/2$ to $x = L/2$. We have also drawn the unit vector $\hat{r}$ that points from the piece of the rod to the test charge. *Note that the unit vector's direction changes as we slide the piece dx from one end of the rod to the other.* This is a *very* important thing to note: the unit vector $\hat{r}$ is a *function*, not just a constant. With trigonometry, it is simple to work out what this function is:

$$\hat{r} = -\hat{x}\sin\theta - \hat{y}\cos\theta\ .$$

With a bit more trigonometry, we can write this as a function of x, the location of the chunk, rather than of θ:

$$\hat{r} = -\hat{x}\frac{x}{\sqrt{a^2 + x^2}} - \hat{y}\frac{a}{\sqrt{a^2 + x^2}}\ .$$

Notice finally that $\sqrt{a^2 + x^2}$ is the distance from the element dx to the test charge.

Defining $\lambda = Q/L$, the force that the test charge feels is given by a simple integral:

$$\vec{F} = -\int_{-L/2}^{L/2} q\lambda \frac{x\hat{x} + a\hat{y}}{(a^2+x^2)^{3/2}}\, dx\ .$$

The horizontal ($\hat{x}$) component is obviously zero: for every element on the right of the midpoint, there is an element on the left whose force magnitude is equal, but whose horizontal component points in the opposite direction. You can prove this by doing the integral, but it's very healthy to understand the physics of this simple *symmetry argument*.

The remaining integral for the y component is now easy: using

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}\ ,$$

we find

$$\begin{aligned}\vec{F} &= -\frac{q\lambda L}{a\sqrt{a^2+L^2/4}}\hat{y} \\ &= -\frac{qQ}{a\sqrt{a^2+L^2/4}}\hat{y}\ .\end{aligned}$$

This is a typical example of the kinds of integrations we will need to do (though most will be more challenging than this simple case).

1.7 Energy of a system of charges

1.7.1 Work done moving charges

Suppose we have a charge Q sitting at the origin. How much work must we do to move a charge q from radius r_2 to radius r_1? Let us first do this assuming that q moves on a purely radial path. The work that we do is given by integrating the force that *we exert* along this path:

$$W = \int \vec{F}_{\rm us} \cdot d\vec{s}$$

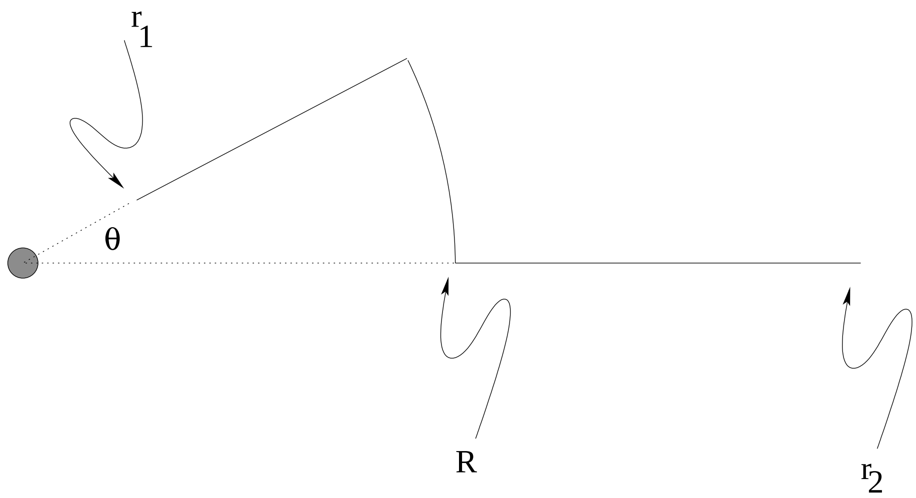
What is $F_{\rm us}$? It is *opposite* to the force that the charge Q exerts on q:

$$\vec{F}_{\rm us} = -\vec{F}_{\rm Coulomb} = -\frac{Qq\hat{r}}{r^2}\ .$$

We integrate this along a radial path, $ds = dr\,\hat{r}$, from $r = r_2$ to $r = r_1$:

$$\begin{aligned}W(r_2 \to r_1) &= \int \vec{F}_{\rm us}\cdot d\vec{s} \\ &= -\int_{r_2}^{r_1} \frac{Qq}{r^2} dr \\ &= \frac{Qq}{r_1} - \frac{Qq}{r_2}\ .\end{aligned}$$

It is straightforward to prove that this result holds along *any* path. Suppose we come in on a radial path to radius R, move in a circular arc through some angle θ, and then continue on a new radial path to r_1:



The differential $d\vec{s}$ in this case is still $dr\,\hat{r}$ along the two radial paths, but it is $R\,d\theta\,\hat{\theta}$ along the angular segment (where $\hat{\theta}$ is a unit vector that points in the angular direction). The integral thus becomes

$$\begin{aligned} W(r_2 \to r_1) &= \int \vec{F}_{\rm us} \cdot d\vec{s} \\ &= -\int_{r_2}^{R} \frac{Qq}{r^2} dr - \int_0^{\theta} \frac{Qq}{R^2} (\hat{r} \cdot \hat{\theta})(R\,d\theta) - \int_R^{r_1} \frac{Qq}{r^2} dr \\ &= \frac{Qq}{R} - \frac{Qq}{r_2} + 0 + \frac{Qq}{r_1} - \frac{Qq}{R} = \frac{Qq}{r_1} - \frac{Qq}{r_2} \; . \end{aligned}$$

We end up with the *exact same result* because the contribution around the arc is zero: that path is perpendicular to the force, so we do no work moving along that segment.

A corollary of this result is that the work we do is the same along *ANY* path that we take from r_2 to r_1. Why? We can break up any path into radial segments and angular arcs, in a similar manner to the above calculation. The angular segments don't matter; the radial segments will give us the same answers we've just derived.

A second corollary is that the electric force is *conservative*: any path that takes us from r_1 back to r_1 does zero work (just plug $r_2 = r_1$ into the above result).

1.7.2 Work done to assemble a system of charges

How much work does it take to *assemble* a system of charges? With the calculations we just did, answering this is easy.

Let's clarify the question a bit: how much work does it take to move a charge q_2 in from infinity until it is a distance r_{12} from a charge q_1? This is *really* what we mean what we ask how much energy it takes to assemble the system — we are asking what it takes to take charges that are very far apart and bring them close together. Using the results we worked out above, the answer is clearly

$$\begin{aligned} W(\infty \to r_{12}) &= \frac{q_1 q_1}{r_{12}} \\ &\equiv W_{12} \; . \end{aligned}$$

This quantity can then be thought of as the energy content U of this system of charges: $U = W$.

Now, how about if we bring in a third charge q_3? This turns out to be easy using the principle of superposition. To make it very clear what is going on, let's go back to the integral we have to do for the work: we bring the charge q_3 in from infinite distance until it is a distance r_{13} from charge q_1 and r_{23} from charge q_2:

$$\begin{aligned} W_{(1+2)3} &= -\int_\infty^{r_{13}} \frac{q_1 q_3}{r^2} dr - \int_\infty^{r_{23}} \frac{q_2 q_3}{r^2} dr \\ &= \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \\ &= W_{13} + W_{23} \ . \end{aligned}$$

We add this to the work it took to bring q_1 and q_2 together to get the total work to assemble this system:

$$\begin{aligned} W &= W_{12} + W_{13} + W_{23} \\ &= \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \ . \end{aligned}$$

Again, this is the total energy content of the system: $U = W$.

We can generalize this to as many charges as we want; the resulting formulas rapidly get pretty big. The key thing to note is that we are summing over *pairs* of charges in the system. It is pretty simple to write down a formula that sums up the energy over pairs like this:

$$U = \frac{1}{2} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ j \neq k}}^{N} \frac{q_j q_k}{r_{jk}}$$

The requirement that $j \neq k$ makes sure that we only count over pairs of charges — if $k = j$, we get nonsense in the formula. The factor of 1/2 out front is needed because the sums will actually count each pair twice. (Try it out for a simple case, $N = 2$ or $N = 3$, to see.)