

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
 DEPARTMENT OF PHYSICS
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LECTURE 3:
 ELECTRIC FIELD ENERGY.
 POTENTIAL; GRADIENT.
 GAUSS'S LAW REVISITED; DIVERGENCE.

3.1 Energy in the field

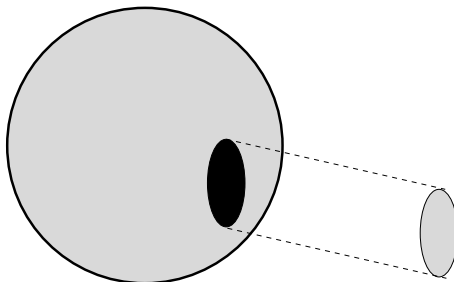
Suppose we take our spherical shell of radius r , with charge per unit area $\sigma = q/4\pi r^2$, and squeeze it. How much work does it take to do this squeezing? To answer this, we need to know how much pressure is being exerted by the shell's electric field on itself. Pressure is force/area; σ is charge/area; force is charge times electric field. Hence, our intuitive guess is that the pressure should be

$$\begin{aligned} P_{\text{guess}} &= (\text{electric field})(\text{charge/area}) \\ &= 4\pi\sigma^2. \end{aligned}$$

(We have used the fact that the electric field just at the surface of the shell is $4\pi\sigma$.)

This is *almost* right, but contains a serious flaw. To understand the flaw, note that the electric field we used here, $E = 4\pi\sigma$, includes contributions from *all* of the charge on the sphere — including the charge that is being acted upon by the field. We must therefore be overcounting in some sense — a chunk of charge on the sphere cannot act upon itself.

It's not too hard to fix this up. Imagine pulling a little circular disk out of the sphere:



The *correct* formula for the pressure will be given by

$$\begin{aligned} P &= (\text{electric field in hole})(\text{charge/area on disk}) \\ &= (\text{electric field in hole})\sigma. \end{aligned}$$

We just need to figure out the electric field in the hole. Fortunately, there's a clever argument that spares us the pain of a hideous integral: we know that the field of the hole *plus* the field of the disk gives us the field of the sphere:

$$\begin{aligned} E_{\text{hole}} + E_{\text{disk}} &= E_{\text{sphere}} \\ &= 4\pi\sigma \quad \text{Just outside the sphere} \\ &= 0 \quad \text{Just inside the sphere.} \end{aligned}$$

What's E_{disk} ? Fortunately, we only need to know this *very* close to the disk itself: from the calculations we did in Lecture 2, we have

$$\begin{aligned} E_{\text{disk}} &= +2\pi\sigma && \text{Outer side of disk} \\ &= -2\pi\sigma && \text{Inner side of disk.} \end{aligned}$$

We're now done:

$$E_{\text{hole}} = 2\pi\sigma \quad (\text{Pointing radially outward})$$

which means that the pressure is

$$P = 2\pi\sigma^2 .$$

(Purcell motivates the factor of 1/2 difference by pointing out that one gets the pressure by *averaging* the field at r over its discontinuity — the field is 0 at radius $r - \epsilon$, it is $4\pi\sigma$ at $r + \epsilon$. This argument is totally equivalent, and is in fact based upon Purcell problem 1.29.)

Now that we know the pressure it is easy to compute the work it takes to squeeze:

$$\begin{aligned} dW &= F dr \\ &= P 4\pi r^2 dr \\ &= 2\pi\sigma^2 dV . \end{aligned}$$

On the last line, we've used the fact that when we compress the shell by dr , we have squeezed out a volume $dV = 4\pi r^2 dr$. Now, using $E = 4\pi\sigma$, we can write this formula in terms of the electric field:

$$dW = \frac{E^2}{8\pi} dV .$$

The quantity $E^2/8\pi$ is the energy density of the electric field itself. (In SI units, we would have found $\epsilon_0 E^2/2$.) It shows up in this work calculation because when we squeeze the shell we are *creating* new electric field: the shell from $r - dr$ to r suddenly contains field, whereas it did not before. We had to put $dW = (\text{energy density}) dV$ amount of work into the system to make that field.

3.2 Electric potential difference

I move a charge q around in an electric field \vec{E} . The work done *by me* as I move this charge from position \vec{a} to \vec{b} is given by

$$\begin{aligned} U_{ba} &= \int_{\vec{a}}^{\vec{b}} \vec{F}_{\text{me}} \cdot d\vec{s} \\ &= - \int_{\vec{a}}^{\vec{b}} \vec{F}_q \cdot d\vec{s} \\ &= -q \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{s} . \end{aligned}$$

\vec{F}_{me} is the force that I exert. By Newton's 3rd, this is equal and opposite to the force F_q that the electric field exerts on the charge. This is why we flip the sign going from line 1 to line

2. Make sure you understand this logic — that sign flip is an important part of stuff we’re about to define! On the last line, we’ve expressed this in terms of the electric field. Recall that one of the things we learned about electric fields is that they generate *conservative* forces. This means that this integral must be independent of the path we take from \vec{a} to \vec{b} . We’ll return to this point later.

Since the work we do is proportional to the charge, we might as well divide it out. Doing so, we define the *electric potential difference* between \vec{a} and \vec{b} :

$$\phi_{ba} = - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{s}.$$

This function has a simple physical interpretation: it is the work that \mathbf{I} must do, *per unit charge*, to move a charge from \vec{a} to \vec{b} . Its units are energy/charge: erg/esu in cgs, Joule/Coulomb in SI. Both of these combinations are given special names: the cgs unit of potential is the *statvolt*, the SI unit is the *Volt*.

3.3 Electric potential

Let’s say we fix the position \vec{a} and consider the electric potential difference between \vec{a} and a variety of different points \vec{r} . Since the point \vec{a} is fixed, we can speak of the *electric potential* $\phi(\vec{r})$ relative to the *reference point* \vec{a} . The language we use here is very important — “potential” (as opposed to “potential difference”) only makes sense provided we know what is the reference point. All potentials are then defined *relative to* that reference point.

In a large number of cases, it is very convenient to take the reference point \vec{a} to be at infinity. (We’ll discuss momentarily when it is not convenient to do so.) In this case, the electric potential at \vec{r} is

$$\phi(\vec{r}) = - \int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{s}.$$

Suppose \vec{E} arises from a point charge q at the origin:

$$\begin{aligned} \phi(\vec{r}) &= - \int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{s} \\ &= - \int_{\infty}^r \frac{q}{r^2} dr \\ &= \frac{q}{r}. \end{aligned}$$

Note that using this formula the potential difference has a very simple form:

$$\begin{aligned} \phi_{ba} &= - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{s} \\ &= \frac{q}{b} - \frac{q}{a} \\ &= \phi(\vec{b}) - \phi(\vec{a}). \end{aligned}$$

As the name suggests, the potential difference is indeed just the difference in the potential between the two points.

By invoking superposition, we can generalize these formulas very simply: Suppose we have N point charges. Using infinity as our reference point, the potential at some point P is given by

$$\phi(P) = \sum_{i=1}^N \frac{q_i}{r_i},$$

where r_i is the distance from P to the i th charge. We can further generalize this to continuous distributions: for a charge density ρ distributed through a volume V ,

$$\phi(P) = \int_V \frac{\rho dV}{r},$$

where r is the distance from a volume element dV to the field point; for surface charge density σ distributed over an area A ,

$$\phi(P) = \int_A \frac{\sigma dA}{r};$$

for a line charge density λ distributed over a length L ,

$$\phi(P) = \int_L \frac{\lambda dl}{r}.$$

3.4 An important caveat

There is one *massive* caveat that must be borne in mind when using these formulas: **they ONLY work if it is OK to set the reference point to infinity**. That, in turn only works if the charge distribution *itself* is of finite size. As a counterexample, consider an infinite line charge, with charge per unit length λ . Using Gauss's law, it is simple to show that the electric field of this charge distribution is given by

$$\vec{E} = \frac{2\lambda}{r} \hat{r},$$

where \hat{r} points out from the line charge, and r is the distance to the line. Let's first work out the potential difference between two points r_1 and r_2 away from the line:

$$\begin{aligned} \phi_{21} &= - \int_{r_1}^{r_2} \vec{E} \cdot d\vec{s} \\ &= -2\lambda \int_{r_1}^{r_2} \frac{dr}{r} \\ &= 2\lambda \ln(r_1/r_2). \end{aligned}$$

Suppose we make r_1 our reference point. Now, look what happens as we let this reference point approach infinity — the potential diverges! For a charge distribution like this, the field does not fall off quickly enough as a function of distance for infinity to be a useful reference point. In cases like this, we need to pick some other reference point. In such a situation, it is crucial that you make clear what you have picked.

3.5 Energy of a charge distribution

If we can use infinity as a reference point, then we can write down a formula for the energy of a charge distribution that is sometimes quite handy. From an earlier lecture, we worked out the formula

$$U = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{q_i q_j}{r_{ij}} .$$

A useful way to rearrange this formula is as follows:

$$\begin{aligned} U &= \frac{1}{2} \sum_i q_i \sum_{j \neq i} \frac{q_j}{r_{ij}} \\ &= \frac{1}{2} \sum_i q_i \phi(r_i) . \end{aligned}$$

On the second line, we have used the fact that the quantity under the second sum is just the electric potential of all the charges *except* q_i , evaluated at q_i 's location.

Now, we take a continuum limit: we put $q_i \rightarrow \rho dV$ and let the sum become an integral. We find

$$U = \frac{1}{2} \int \rho \phi dV .$$

This formula is *entirely* equivalent to other formulas we worked out for the energy of a charge distribution. Bear in mind, though, that we have implicitly assumed that the potential goes to zero infinitely far from the charge distribution!

3.6 Field from the potential

Consider the potential difference between the point \vec{r} and $\vec{r} + \Delta\vec{r}$:

$$\begin{aligned} \Delta\phi &= - \int_{\vec{r}}^{\vec{r} + \Delta\vec{r}} \vec{E} \cdot d\vec{s} \\ &\simeq - \vec{E}(\vec{r}) \cdot \Delta\vec{r} . \end{aligned}$$

Now, expand the vectors in Cartesian coordinates:

$$\begin{aligned} \Delta\phi &\simeq -(E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (\Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z}) \\ &\simeq -E_x \Delta x - E_y \Delta y - E_z \Delta z . \end{aligned}$$

Dividing, taking the limit $\Delta \rightarrow 0$, and invoking the definition of the partial derivative, we find

$$E_x = -\frac{\partial\phi}{\partial x} , \quad E_y = -\frac{\partial\phi}{\partial y} , \quad E_z = -\frac{\partial\phi}{\partial z} ;$$

or

$$\begin{aligned} \vec{E} &= -\hat{x} \frac{\partial\phi}{\partial x} - \hat{y} \frac{\partial\phi}{\partial y} - \hat{z} \frac{\partial\phi}{\partial z} \\ &\equiv -\vec{\nabla}\phi . \end{aligned}$$

On the last line we have defined the *gradient operator*,

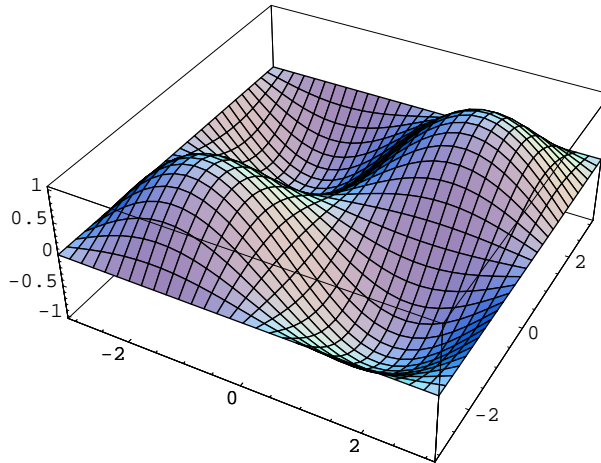
$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} .$$

The electric field is thus given by the *gradient* of the potential. You will sometimes see this written $\vec{E} = -\mathbf{grad} \phi$.

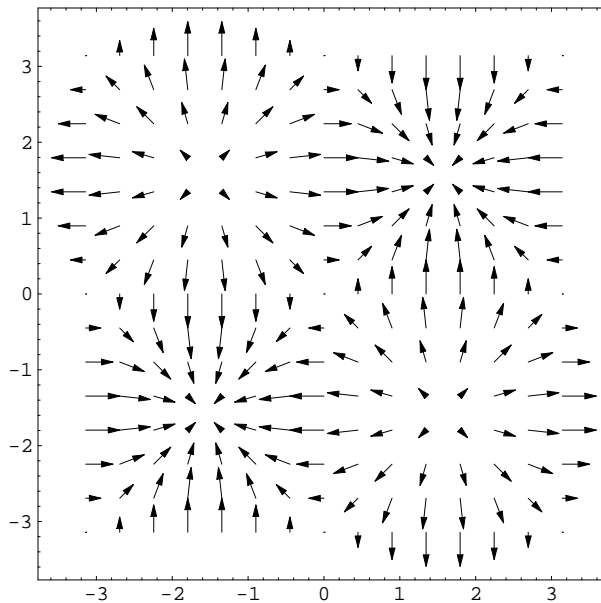
3.7 The gradient and equipotentials

We've defined a mathematical operation which we named the gradient. What exactly *is* it? To get some intuition, let's imagine a one dimensional potential, $\phi(x)$. The gradient is given by $\vec{\nabla}\phi = \hat{x} d\phi/dx$. Bearing in mind that the derivative is the function's *slope*, the gradient appears to tell us the direction in which the function changes, and the rate of that change.

A 1-D example is simple, but also a touch silly. Let's try a 2-D example: put $\phi(x, y) = \sin(x) \sin(y)$. This potential looks like



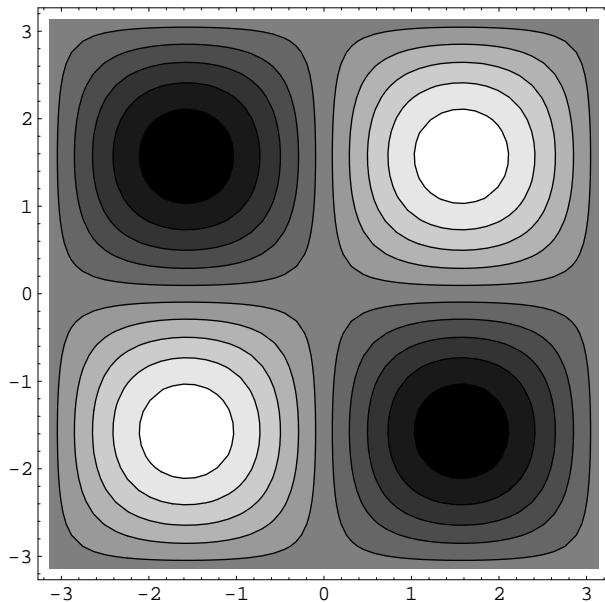
It is simple to calculate this potential's gradient: $\vec{\nabla}\phi = \hat{x} \cos(x) \sin(y) + \hat{y} \sin(x) \cos(y)$. This vector field looks like



Notice the way the little arrows are pointing. Our intuition from the silly 1-D example is exactly right: the gradient points in the direction in which the function increases fastest. At the minima (upper left, lower right), the gradient points out. At the maxima (upper right, lower left), the gradient points in. A simple way to summarize this

is to remember that the gradient always points “uphill”. Bearing in mind that the electric field is the *negative* gradient of ϕ , we can say that the electric field points “downhill”.

Another useful concept we can build from these plots is the notion of an *equipotential surface*. As the name suggests, an equipotential surface is just a surface along which the potential takes some constant value. For our 2-D example, the equipotential “surfaces” are really curves in the $x - y$ plane:



We can deduce one further important property of equipotential surfaces. By definition, the potential is constant on them, so they define the directions along which the potential does not change at all. The gradient, by contrast, defines the direction along which the potential changes *the most!* If you think about this for a moment, it becomes clear the gradient must be exactly perpendicular to equipotential surfaces. Since the electric field is $-\vec{\nabla}\phi$, it follows that the electric field is also exactly perpendicular to equipotentials. A very handy way of putting this is to say that electric field lines are orthogonal to equipotential surfaces.

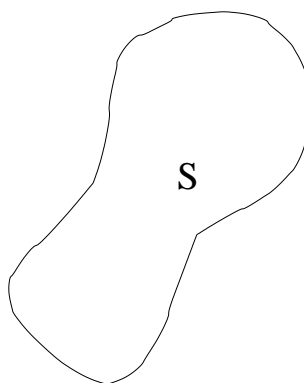
(If you are a hiker and you have ever used a topographic map, much of this should seem quite familiar. The contours of a topo map label surfaces of constant height, which is exactly related to the *gravitational* potential. If you move perpendicular to the contour lines, you are moving along the steepest slope possible — desperately scrambling up, or rather quickly tumbling down.)

3.8 Divergence

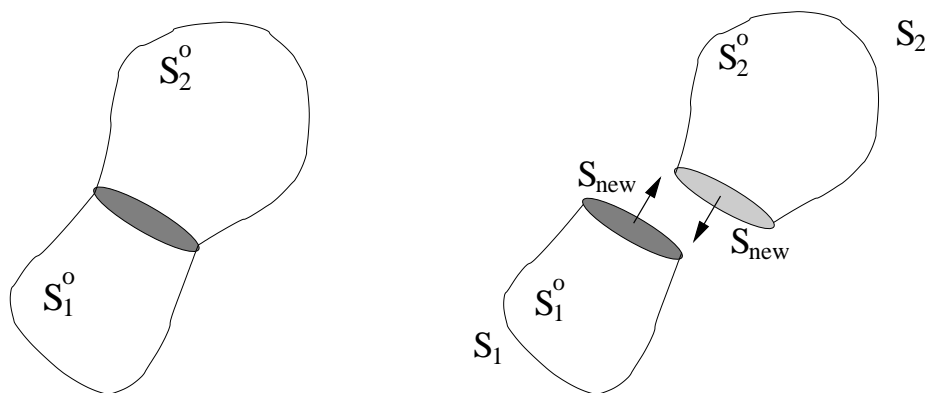
The gradient which we just learned about is the first and simplest mathematical operation in *vector calculus*. We’re going to continue this theme for a little while, exploring several vector calculus operations which we will use exhaustively in this course.

To set up the next vector calculus operation, let’s return to the notion of flux. Suppose we’ve got some vector field \vec{F} , and we want to calculate its flux Φ through a closed surface S :

$$\Phi = \oint_S \vec{F} \cdot d\vec{A}.$$



Now, we do something that may seem kind of silly: we cut the volume into two pieces.



(For clarity, the two halves on the right are drawn separated. We really leave the two pieces right next to each other.) Before calculating, let's make sure we understand what the various surfaces in this diagram mean. On the left, we have the surfaces S_1^o and S_2^o . These are just the top-right and bottom-left halves of the "old" surface S . Note that these are *not closed* surfaces. On the right, we still have S_1^o and S_2^o ; in addition, we have a pair of surfaces labeled S_{new} . These surfaces are exactly the same area but are oriented in opposite directions. S_1 and S_2 are the *closed* surfaces surrounding the two halves of the blob.

OK, so what's the flux in this case? It must be exactly the same, Φ , since everything's still the same as far as the field \vec{F} is concerned. But the way we calculate it is different:

$$\begin{aligned}
 \Phi &= \oint_S \vec{F} \cdot d\vec{A} \\
 &= \int_{S_1^o} \vec{F} \cdot d\vec{A} + \int_{S_2^o} \vec{F} \cdot d\vec{A} \\
 &= \int_{S_1^o} \vec{F} \cdot d\vec{A} + \int_{S_2^o} \vec{F} \cdot d\vec{A} + \int_{S_{\text{new}}} \vec{F} \cdot d\vec{A} - \int_{S_{\text{new}}} \vec{F} \cdot d\vec{A} \\
 &= \left(\int_{S_1^o} \vec{F} \cdot d\vec{A} + \int_{S_{\text{new}}} \vec{F} \cdot d\vec{A} \right) + \left(\int_{S_2^o} \vec{F} \cdot d\vec{A} + \int_{S_{\text{new}}} \vec{F} \cdot (-d\vec{A}) \right) \\
 &= \oint_{S_1} \vec{F} \cdot d\vec{A} + \oint_{S_2} \vec{F} \cdot d\vec{A} \\
 &\equiv \Phi_1 + \Phi_2 .
 \end{aligned}$$

Let's go through this carefully. On the second line, we have just broken the closed surface S into the two open surfaces S_1° and S_2° . On the third line, we have added and subtracted an integral over S_{new} . On the fourth line, we've gathered these terms together in a special way. This special way shows that the combined surfaces are just the *closed* surfaces S_1 and S_2 . On the sixth line, we define these two integrals as Φ_1 and Φ_2 .

We can keep dividing this volume up into smaller and smaller subvolumes; everytime we do so, there will be facing surfaces whose orientations will cancel each other out. This allows us to break the total flux up into contributions that come from the surfaces of all these little subvolumes:

$$\Phi = \sum_{i=1}^{\text{big}} \Phi_k .$$

Since we can keep chopping it up into little volumes, the total flux should be proportional to the volume. This motivates the definition of the *divergence*:

$$\mathbf{div} \vec{F} = \lim_{V \rightarrow 0} \frac{\oint_S \vec{F} \cdot d\vec{A}}{V} .$$

In words, the divergence is what you get when you compute the flux through the surface S bounding a volume V , *divided by* V , in the limit in which V goes to zero.

3.9 Gauss's theorem and Gauss's law

Now that we have defined the divergence, we are set to prove a very important theorem. We start with this equation again,

$$\Phi = \sum_{i=1}^{\text{big}} \Phi_k ,$$

but now we write out the integrals:

$$\begin{aligned} \int_S \vec{F} \cdot d\vec{A} &= \sum_{i=1}^N \oint_{S_i} \vec{F} \cdot d\vec{A}_i \\ &= \sum_{i=1}^N V_i \frac{\oint_{S_i} \vec{F} \cdot d\vec{A}_i}{V_i} \\ &= \sum_{i=1}^N V_i \mathbf{div} \vec{F} \\ &= \int_V \mathbf{div} \vec{F} dV . \end{aligned}$$

In words, the flux of a vector function \vec{F} through a closed surface S equals the integral of $\mathbf{div} \vec{F}$ over the volume enclosed by S . This is called *Gauss's theorem*.

As a concrete example, let's take \vec{F} to be the electric field \vec{E} . We thus have

$$\oint_S \vec{E} \cdot d\vec{A} = \int_V \mathbf{div} \vec{E} dV .$$

However, we proved a while ago that

$$\begin{aligned}\oint_S \vec{E} \cdot d\vec{A} &= 4\pi Q(\text{inside } S) \\ &= 4\pi \int_V \rho dV .\end{aligned}$$

Combining these, we have

$$\int_V \mathbf{div} \vec{E} dV = 4\pi \int_V \rho dV ,$$

or

$$\int_V (\mathbf{div} \vec{E} - 4\pi\rho) dV = 0 .$$

This equation is only true for all volumes V only if

$$\boxed{\mathbf{div} \vec{E} = 4\pi\rho}$$

This equation also expresses Gauss's law, only in differential (rather than integral) form. Learn it well! In SI units, in keeping with the rule that the electric field carries a factor $1/4\pi\epsilon_0$, this formula becomes

$$\mathbf{div} \vec{E} = \rho/\epsilon_0 \quad (\text{SI}) .$$