

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.022 SPRING 2005

LECTURE 5:
FIELDS AND POTENTIALS AROUND CONDUCTORS
THE ELECTROSTATIC UNIQUENESS THEOREM

5.1 Conductors

For much of the rest of this class, we will be concerned with processes that involve *conductors*: materials in which it is easy for charges to move around. We will discuss conductors in some depth when we discuss currents; for now, we will just summarize a few of their properties.

Among the best conductors are metals — silver, gold, copper, aluminum, etc. The atoms of these metals form a crystalline structure in which electrons can easily hop around from atom to atom. Although a chunk of metal is neutral overall, we can visualize it as being made of lots of positive charges that are nailed in place, paired up with lots of negative charges (electrons) that are free to move around. In isolation, the negative charges will sit close to the positive charges, so that the metal is not only neutral overall, but also largely neutral everywhere (no *local* excess of positive or negative charge). Under the influence of some external field, the electrons are free to move around.

Some materials conduct OK, but are not as good as metals. For example, salty water has lots of charges that are free to move around under the influence of an \vec{E} field. However, since these charges — usually sodium and chlorine ions — are *far* more massive than an electron, and they do not flow in a crystalline structure, salty water is not a very high quality conductor. Another example is graphite: the somewhat unusual bond structure of graphite makes it a fairly good conductor, but only in certain directions.

At the opposite end of the spectrum are *insulators*: materials in which the electrons are bound quite tightly to the constituent molecules and hence have essentially no freedom to move. Insulators are typically made from organic materials such as rubber or plastic, or from crystals formed from strong covalently bounded molecules, such as quartz or glass.

The effectiveness of a substance as a conductor is quantified by its *resistivity*, a number that expresses how well it resists the flow of current. We will revisit this quantity in about a week when we discuss currents in detail; for now, suffice to say that small resistivity means a good conductor. In SI units, resistivity is measured in “Ohm-meters” (usually written $\Omega\text{-m}$); in cgs units, resistivity turns out to be measured in seconds. Here are a few example values:

Material	Resistivity ($\Omega\text{-m}$)	Resistivity (sec)
Silver	1.6×10^{-8}	1.8×10^{-17}
Copper	1.7×10^{-8}	1.9×10^{-17}
Gold	2.4×10^{-8}	2.6×10^{-17}
Iron	1.0×10^{-7}	1.1×10^{-16}
Sea water	0.2	2.2×10^{-10}
Polyethylene	2.0×10^{11}	220
Glass	$\sim 10^{12}$	$\sim 10^3$
Fused quartz	7.5×10^{17}	8.3×10^8

As you can see, the resistivity of ordinary materials varies over an enormous range, reflecting the very different electronic properties of materials around us.

5.2 Electric fields and conductors

For the rest of this lecture, we will assume that conductors are materials that have an infinite supply of charges that are free to move around. (This of course just an idealization; but, it turns out to be an extremely good one. Real conductors in fact behave very similar to this limit.) From this, we can deduce a few important facts about conductors and electrostatic fields:

- **There is no electric field inside a conductor.** Why? Suppose we bring a plus charge near a conductor. For a very short moment, there will be an electric field inside the conductor. However, this field will act on and move the electrons, which are free to move about. The electrons will move close to the plus charge, leaving net positive charge behind. The conductor's charges will continue to move until the "external" \vec{E} -field is cancelled out — at that point there is no longer an \vec{E} -field to move them, so they stay still.

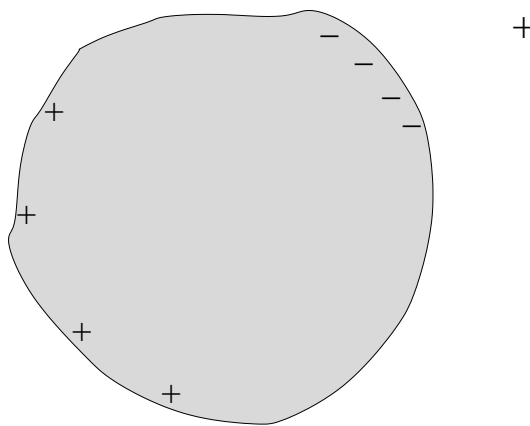


Figure 1: Conductor near an external charge. The charges in the conductor very quickly rearrange themselves to cancel out the external field.

A more accurate statement of this rule is “**After a very short time, there is no electric field inside a conductor**”. How short a time is it? Recall that in cgs units,

resistivity (which tells us how good/bad something conducts electricity) is measured in seconds. It turns out that the time it takes for the charges to rearrange themselves to cancel out the external \vec{E} -field is just about equal to this resistivity. For metals, this is a time that is something like $10^{-16} - 10^{-17}$ seconds. This is so short that we can hardly complain that the original statement isn't precise enough!

- **The electric potential within a conductor is constant.** Proof: the potential difference between any two points \vec{a} and \vec{b} inside the conductor is

$$\begin{aligned}\phi_b - \phi_a &= - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{s} \\ &= 0\end{aligned}$$

since $\vec{E} = 0$ inside the conductor. Hence, for any two points \vec{a} and \vec{b} inside the conductor, $\phi_b = \phi_a$.

- **Net charge can only reside on the surface of a conductor.** This is easily proved with Gauss's law: make a little Gaussian surface that is totally contained inside the conductor. Since there is no \vec{E} -field inside the conductor, $\oint \vec{E} \cdot d\vec{A}$ is clearly zero for your surface. Since that is equal to the charge the surface contains, there can be no charge.

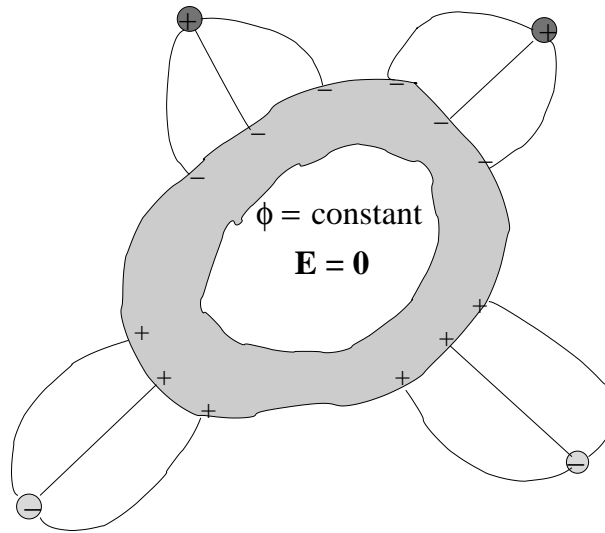
We will discuss the charge on the conductor's surface in a moment.

- **Any external electric field lines are perpendicular to the surface.** Another way to put this is that there is no component of electric field that is tangent to the surface. We prove this by contradiction: suppose that a component of the \vec{E} -field were tangent to the surface. If that were the case, then charges would flow along the surface. They would continue to flow until there was no longer any tangential component to the \vec{E} -field. Hence, this situation cannot exist: even if it exists momentarily, it will rapidly (within 10^{-17} seconds or so) correct itself.
- **The conductor's surface is an equipotential.** This follows from the fact that the \vec{E} -field is perpendicular to the surface. We do a line integral of \vec{E} on the surface; the path is perpendicular to the field; so the difference in potential between any two points on the surface is zero.

A few important corollaries follow from these rules.

- **Corollary 1:** Consider a conductor with a hollowed out region. If there is no charge in this hollow, then the potential ϕ there is constant. It follows that the electric field $-\vec{\nabla}\phi$ inside the hollow is zero.

We will prove this carefully shortly. For now, we can motivate this proof by noting that the surface of the conductor must be an equipotential. Since there is no charge anywhere on the inside, the interior potential must obey Laplace's equation, $\nabla^2\phi = 0$. Solutions to Laplace's equation can have no local maxima or minima. The only solution that has some proscribed constant value on an exterior boundary *and* has no local maxima or minima is one that is constant.



By putting things inside a conducting box, it is shielded from any external electric field. Such an arrangement is known as a Faraday cage. Although it is beyond the scope of this course to prove this, Faraday cages work pretty well even when the external fields vary in time, as long as they don't vary too quickly.

- **Corollary 2:** suppose we put a charge q inside this hollow. There must be an *induced* layer of surface charge on the wall of the hollow; the total amount of induced charge in this layer is $q_{\text{ind}} = -q$.

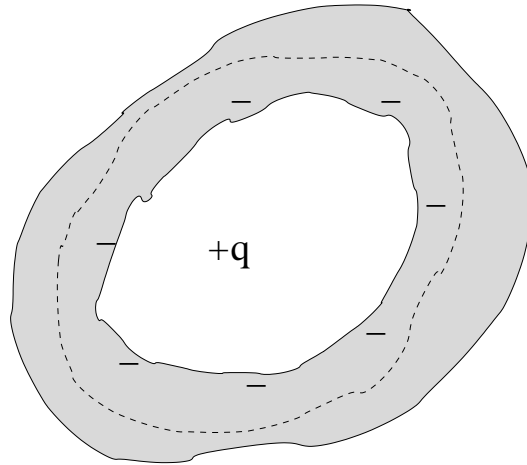


Figure 2: Gaussian surface: dashed line enclosing hollow and induced negative charge.

To prove this, use Gauss's law. We draw a Gaussian surface that lies inside the conductor. The \vec{E} -field there is zero, so the total electric flux must be zero. This flux equals 4π times the total charge contained by the surface, which is q plus the induced charge:

$$\begin{aligned} \oint \vec{E} \cdot d\vec{A} &= 0 \\ &= 4\pi(q + q_{\text{ind}}) \\ \rightarrow q_{\text{ind}} &= -q. \end{aligned}$$

You should be able to prove with another well chosen Gaussian surface that the conductor has another layer of charge on its *outer* surface with total magnitude $+q$.

- **Corollary 3:** The induced charge *density* on any surface is given by $|\vec{E}|/4\pi$, where \vec{E} is the electric field right next to the surface. This follows from the rule that whenever there is a surface charge layer, the electric field changes with $\Delta E = 4\pi\sigma$. Since the field inside the conductor is zero, $\Delta E = |\vec{E}|$.

5.3 The uniqueness theorem

Suppose we know that some region contains a charge density $\rho(\vec{r})$. Suppose we also know the value of the electrostatic potential $\phi(\vec{r})$ value on the boundary of this region. The *uniqueness theorem* then guarantees that there is only one function $\phi(\vec{r})$ which describes the potential in that region. This means that no matter how we figure out ϕ 's value — guessing, computer aided numerical computation, demonic invocation — the function ϕ we find is *guaranteed* to be the one we want.

We prove this theorem by assuming it is not true: we assume that two functions, $\phi_1(\vec{r})$ and $\phi_2(\vec{r})$, both satisfy Poisson's equation:

$$\begin{aligned}\nabla^2\phi_1(\vec{r}) &= -4\pi\rho(\vec{r}) \\ \nabla^2\phi_2(\vec{r}) &= -4\pi\rho(\vec{r}) .\end{aligned}$$

Both of these functions must satisfy the boundary condition: we must have $\phi_1(\vec{r}) = \phi_2(\vec{r}) = \phi_B(\vec{r})$ on the boundary of our region.

By the principle of superposition, any combination of the potentials $\phi_1(\vec{r})$ and $\phi_2(\vec{r})$ must be a perfectly valid potential. (It's not the potential that describes our region, but it still is valid as far as the general laws of physics are concerned.) Let's look in particular at $\phi_3(\vec{r}) = \phi_2(\vec{r}) - \phi_1(\vec{r})$. First, what is the boundary condition for ϕ_3 ? Since $\phi_1 = \phi_2$ on the boundary, we must have $\phi_3 = 0$ there. Next, apply the Laplacian to ϕ_3 :

$$\begin{aligned}\nabla^2\phi_3 &= \nabla^2(\phi_2 - \phi_1) \\ &= \nabla^2\phi_2 - \nabla^2\phi_1 \\ &= 4\pi\rho - 4\pi\rho = 0 .\end{aligned}$$

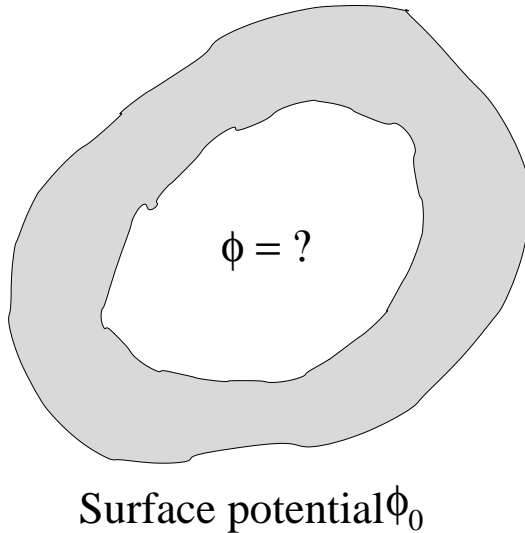
The potential ϕ_3 thus satisfies Laplace's equation. This means that it can have no local maxima or minima inside its boundary. But, on the boundary, its value is zero! The only function which is zero on a boundary **and** has no local maxima or minima is one which is zero *everywhere* in the region: $\phi_3(\vec{r}) = 0$. This means that $\phi_1(\vec{r}) = \phi_2(\vec{r})$ — our initial assumption, that at least two potentials satisfied Poisson's equation in the region with the given boundary condition, was wrong. Hence the solution for $\phi(\vec{r})$ is unique. Putting it concisely, we have proved the uniqueness theorem: **An electrostatic potential $\phi(\vec{r})$ is completely determined within a region once its value is known on the region's boundary.**

This is good news: it gives us license to be lazy.

5.4 Some conductor examples

5.4.1 Example 1: Randomly shaped conductor

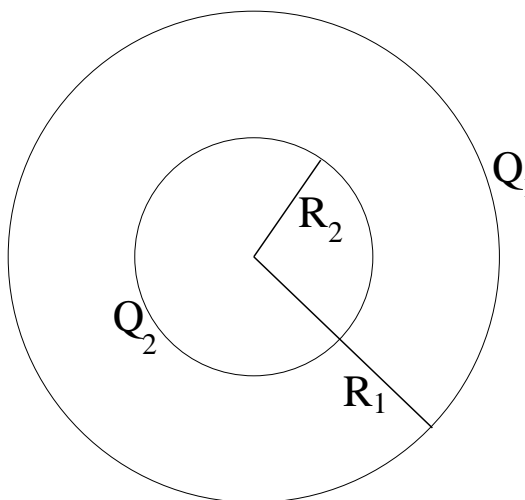
A chunk of conductor with a hollowed out core has charge placed on it until its potential (relative to infinity) is ϕ_0 . What is the potential inside the core?



Answer: the potential inside the core — indeed, everywhere inside the conductor's surface — must be ϕ_0 . Why? $\phi(\vec{r}) = \phi_0$ is a solution that satisfies the boundary condition, and it obviously satisfies Laplace's equation (since ϕ_0 is a constant). By the uniqueness theorem, that's all we need.

5.4.2 Example 2: Nested concentric spherical shells

A pair of thin, conducting, nested spherical shells with radii R_2 and R_1 carry charges Q_2 and Q_1 respectively. What is the potential of the inner sphere? What is the potential of the outer sphere? What is the potential at $r = 0$? (Set the zero of the potential at infinity.)



Let's do the potential of the outer sphere first: it must be

$$\phi_1 = \frac{Q_1 + Q_2}{R_1}.$$

This should hopefully be almost obvious — if not, apply Gauss's law outside the outer sphere and note that it looks just like a point charge of $Q_1 + Q_2$ at the origin.

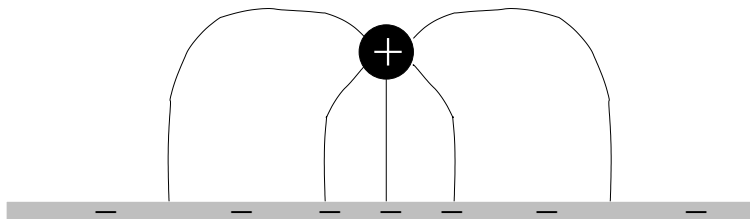
To get the potential of the inner sphere, we use

$$\begin{aligned} \phi_2 - \phi_1 &= - \int_{R_1}^{R_2} \vec{E} \cdot d\vec{s} \\ &= - \int_{R_1}^{R_2} \frac{Q_2}{r^2} dr \\ &= \frac{Q_2}{R_2} - \frac{Q_2}{R_1} \\ \rightarrow \phi_2 &= \frac{Q_2}{R_2} + \frac{Q_1}{R_1}. \end{aligned}$$

By the uniqueness theorem, the potential at $r = 0$ — indeed, for *any* $r \leq R_2$ — must be ϕ_2 .

5.4.3 Example 3: Point charge and flat plane conductor

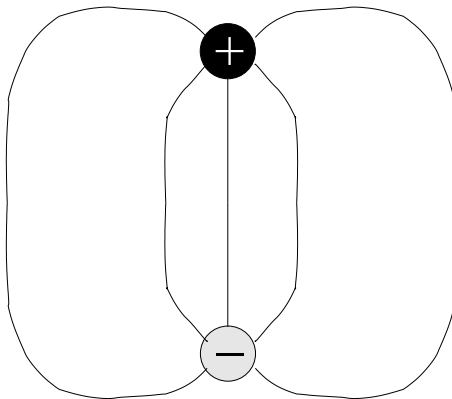
Suppose we hold a point charge Q a distance d above an infinite, flat plane conductor. The potential on the conductor is fixed at zero¹. What is the potential everywhere above the conductor?



A priori, this looks nasty: the point charge induces minus charge on the conductor, which makes things really complicated. What do we do?

¹This is easily done by hooking it up to *ground*. “Ground” effectively means an infinite supply of plus and minus charges. If we define ground as our reference potential, $\phi = 0$, then anything hooked up to it will also be at $\phi = 0$.

Uniqueness theorem to the rescue. It doesn't matter how we find the potential, as long as we do so in a way that satisfies the boundary condition: the potential must be $\phi = 0$ on the conductor, and the electric field must be purely vertical in the conductor's plane. A *different* configuration which produces the *exact same potential* above the plane is the following one:



We've eliminated the plane entirely, and introduced a minus charge a distance d below the conductor's surface. This fictitious charge is called an *image* charge. With this configuration, the plane right between the two charges (where the conductor's surface is located) will be at $\phi = 0$ (superposition: each point is equidistant from both charges, which are of opposite sign). Also the field is purely vertical at that plane (horizontal components cancel by symmetry). The potential satisfies the boundary condition; it satisfies Laplace's equation; therefore, by the uniqueness theorem it is **THE** solution.

The solution is of course complete nonsense in the region below the conductor's surface. But we don't care! That region was not part of the study anyway.