

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
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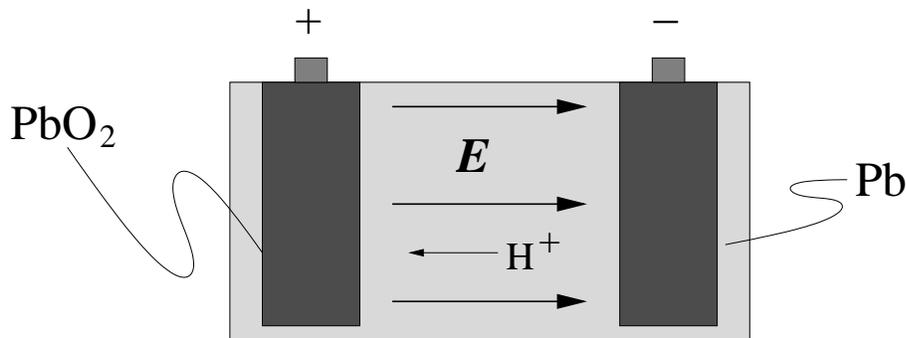
LECTURE 8:
EMF, CIRCUITS, KIRCHHOFF'S RULES

8.1 EMF: Electromotive force

The workings of our technological civilization are largely based on the idea that we use electric currents as a means to get devices to do work for us. In this lecture, we will begin to understand at a fundamental level how this works.

In order for a current to flow, we must create a potential difference over some conductive material; and, we must have continuous source of charges that can flow through our conductor. Such a source, with a “built-in” potential difference is called a source of “electromotive force” (a rather misleading name, since it isn't a force). This is usually abbreviated EMF, and is often denoted \mathcal{E} .

The simplest example of an EMF source is a battery. A battery is a device that maintains a separation of charges between two terminals. Current can flow inside the battery from one terminal to another via an electrochemical reaction. One example, used in many car batteries, is based on the electrochemistry of lead. Two terminals, one of lead oxide, another of porous lead, are immersed in sulphuric acid.



When immersed in the acid, it is energetically favorable for the porous lead terminal to provide free electrons, producing lead sulfate and free hydrogen ions in the solution:



At the lead oxide electrode, a reaction which absorbs free electrons and free hydrogen ions is energetically favorable:



If it is possible for both electrons and H^+ ions to travel from one terminal to the other, then the net chemical reaction



can proceed.

When the terminals of the battery shown above are not connected, there is no way for electrons to get from one terminal to another. An electric field inside the battery builds up, pointing from the + terminal to the - terminal. This field opposes the motion of H^+ ions — they cannot cross to the + terminal, and the reaction stops. When the terminals are connected by a conductor, on the other hand, electrons freely flow to the + terminal. The electric field is reduced, the H^+ ions can now easily move across, and the reaction runs happily.

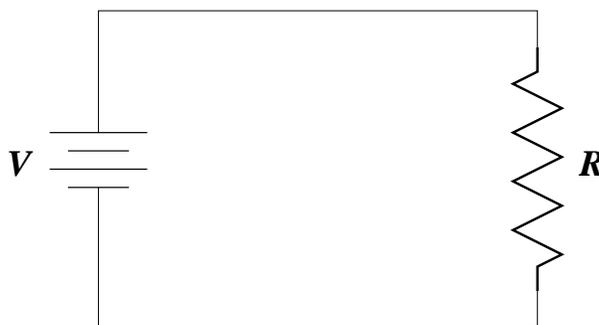
The EMF of the battery is just the potential of the + terminal with respect to the - terminal:

$$\mathcal{E} = - \int_{- \text{ term}}^{+ \text{ term}} \vec{E} \cdot d\vec{s} .$$

This EMF is then the potential difference that is available to drive currents in an electrical circuit.

8.2 Circuits and Kirchhoff's second rule

Suppose we take a battery that provides EMF V and connect its terminals with conducting material of resistance R :



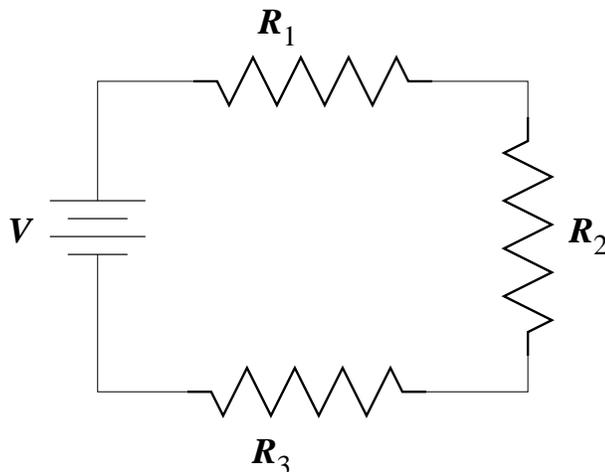
(The symbol on the left is used for a battery in a circuit; the long bar denotes the + terminal, the short bar denotes -. The symbol on the right describes a “resistor”, some conducting material with known resistance R .) How much current flows in this circuit? From Ohm's law, we must have

$$V = IR \rightarrow I = V/R .$$

Another way to view this is to say that when a current I flows through a resistance R , there is a “voltage drop” $\Delta V = -IR$. This motivates what is known as “Kirchhoff's second¹ rule” for circuits: **Around any closed loop, the sum of the EMFs and the potential drops across circuit elements must equal zero.** This rule was first deduced by the German scientist G. Kirchhoff in the mid 1800s; really, it is just a restatement of the rule $\oint \vec{E} \cdot d\vec{s} = 0$.

¹We'll get to Kirchhoff's first rule soon enough.

Suppose we have multiple resistors in our circuit:



The battery still supplies an EMF V to the circuit. Now, it must be shared among all three resistors. To figure out how much voltage drop occurs across each resistor, notice that the current must be the same through all of the circuit elements — if it were not the same, charge would pile up somewhere in the circuit. The voltage drop across the j th resistor is $\Delta V_j = -IR_j$; for the whole circuit, Kirchoff's second rule tells us

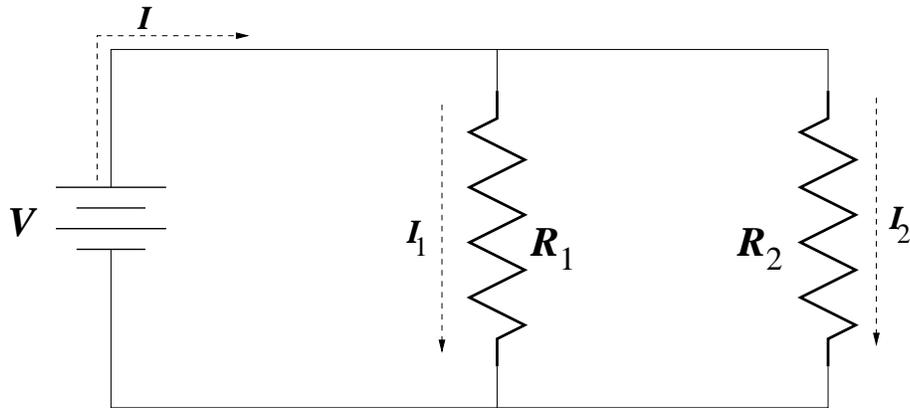
$$\begin{aligned} V + \sum_j \Delta V_j &= 0 \\ V - I(R_1 + R_2 + R_3) &= 0 \\ \rightarrow I &= \frac{V}{R_1 + R_2 + R_3}. \end{aligned}$$

This is equivalent to a circuit containing a *single* resistor, with $R_{\text{eq}} = R_1 + R_2 + R_3$. This logic obviously generalizes: if we have N resistors hooked up in series like this, the equivalent resistance of the circuit is given by

$$R_{\text{eq}} = \sum_{i=1}^N R_i.$$

8.3 Circuits and Kirchhoff's first rule

In the previous example, the resistors were connected in *series*. Another important network topology comes from connecting the resistors in *parallel*:



The battery in this circuit supplies a total current I to all circuit elements. This current divides up into pieces I_1 and I_2 . Kirchhoff #2 tells us that the sum of the EMFs and the voltage drops around *any* closed loop must be zero. A circuit network like that shown here contains three loops to which we can apply this rule: the left loop, the right loop, and the loop that goes around the perimeter of the circuit. The left loop tells us

$$V = I_1 R_1 \rightarrow I_1 = \frac{V}{R_1} ;$$

the perimeter loop tells us

$$V = I_2 R_2 \rightarrow I_2 = \frac{V}{R_2} .$$

(The right loop tells us $I_1 R_1 = I_2 R_2$, which we already knew.)

To make further progress, we need to introduce Kirchhoff's first rule: **The sum of the currents entering any junction must equal the sum of the currents coming out of the junction.** This is a kind of “well, duh” rule: if it were not true, charge would pile up like mad in the circuit, which we know can't happen in the steady state. In the case of this circuit, it tells us that $I = I_1 + I_2$, so

$$I = V \left(\frac{1}{R_1} + \frac{1}{R_2} \right) .$$

The resistors in parallel act like a single resistor with equivalent resistance R_{eq} given by

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} .$$

More generally, if you have N resistors in parallel, their equivalent resistance is

$$\frac{1}{R_{\text{eq}}} = \sum_{i=1}^N \frac{1}{R_i} .$$

In series, multiple resistors add, increasing the net resistance — there is only one path for the current to follow, and you are making that path “harder” by increasing its resistance. In parallel, multiple resistors add reciprocals, *decreasing* the net resistance. Rather than making the path “harder” for the current to follow, you are opening up new paths for it.

In many (though *not* all) cases, one can use the parallel and series rules to reduce a circuit to an “equivalent circuit” that is much simpler: every time you encounter resistors in series, replace them with an equivalent resistor equal to their sum; every time you encounter resistors in parallel, replace them with equivalent resistor whose reciprocal resistance equals the sum of their reciprocal resistance. In some cases, combinations of resistances that look absolutely horrible can be reduced to a single equivalent resistor; see, for example, Figure 4.16 of Purcell.

8.4 Power dissipation in circuits

As current flows in the circuit, it moves through a potential difference. Work must be done for this current to keep flowing. If the source of EMF drives charge dq through the potential difference V , the amount of work that is done is $dW = V dq$. If this is done in time dt , the *rate* at which work is done — i.e., the *power* that is exerted — is given by

$$P = \frac{dW}{dt} = V \frac{dq}{dt} = V I .$$

For a circuit element that obeys Ohm’s law, this tells us that

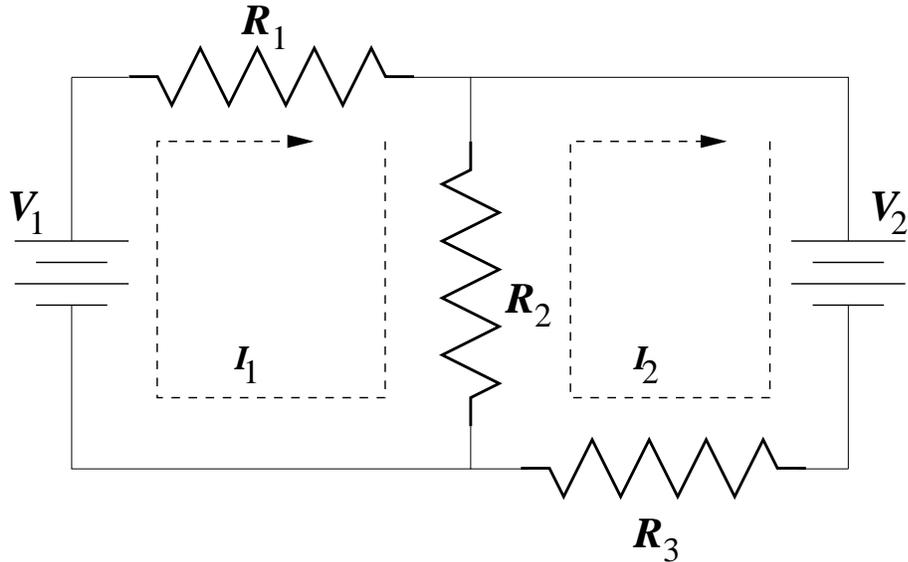
$$P = (IR) I = I^2 R .$$

This is also the power that is *dissipated* in a resistor. If you touch a circuit element and it feels hot, this is because it is dissipating energy at the rate $I^2 R$.

Units: in cgs, I is in esu/sec, and V is in esu/cm. This unit of potential is equivalent to erg/esu — recall the potential difference is work or energy per unit charge. IV thus has units erg/sec. In SI, I is in amps, or Coulomb/sec; V is in Volts, or Joules/Coulomb. IV thus has units Joules/sec, or Watts.

8.5 Multiple voltage source networks

How do we handle the following circuit?



The multiple sources of EMF in this circuit network make this problem somewhat more complicated than those seen so far. In particular, we cannot simply reduce the resistors into those that are in series in the circuit and those that are in parallel — the extra battery makes it hard to even define a meaningful notion of “in series” or “in parallel” in this case.

Kirchhoff’s rules save us, provided we establish procedures for how to use them:

1. Regard the circuit as collection of independent closed current loops. “Independent” means that the loops may overlap, but each loop has at least one portion that does not overlap with other loops. In the example show above, there are two independent loops: the left loop and the right loop are independent (though they share one common leg); the left loop and the whole perimeter are independent; the right loop and the perimeter are independent. Once you’ve picked two, the third can be regarded as a subset of them.
2. Label the currents in the loops, I_1 , I_2 , etc, until each loop has some circulating current assigned to it. Pick the direction arbitrarily; if you guess “wrong”, but do the subsequent calculation correctly, the current will just come out with a minus sign.
3. Apply Kirchhoff’s #2 to each loop: traverse the loop in the direction of the current that you established. Every time you cross a resistor, you get a voltage drop $-IR$; the I to be used here is the *total* current passing through the resistor.

Every time you cross an EMF source, it contributes $+V$ if you cross it in the “right” direction (from the $-$ terminal to the $+$). It contributes $-V$ if you cross it in the “wrong” direction ($+ \rightarrow -$).

4. Apply Kirchhoff’s #1 at each junction.

Following these rules, we end up with a set of linear equations which can be reduced to find all unknown currents in the circuit (albeit sometimes requiring a hideous amount of

algebra). Let's look at the circuit above as an example: Traversing the left loop, Kirchhoff #2 tells us

$$V_1 - I_1 R_1 - (I_1 - I_2) R_2 = 0 .$$

For the right loop, we find

$$-V_2 - I_2 R_3 - (I_2 - I_1) R_2 = 0 .$$

Kirchhoff's #1 is automatically satisfied, since we have $I_1 + I_2$ both entering and leaving each junction. Rearranging these equations yields

$$\begin{aligned} V_1 &= I_1 R_1 + (I_1 - I_2) R_2 \\ V_2 &= -I_2 R_3 - (I_2 - I_1) R_2 . \end{aligned}$$

With a little effort, we can reduce these equations to find

$$\begin{aligned} I_1 &= \frac{V_1 R_3 + (V_1 - V_2) R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} , \\ I_2 &= \frac{(V_1 - V_2) R_2 - V_2 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3} . \end{aligned}$$

Although it is somewhat beyond the scope of this course, it is worth noting that the equations we get with Kirchhoff's laws can be rewritten as a matrix equation. For the example discussed here, we would find

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 & -R_2 \\ R_2 & -R_2 - R_3 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} .$$

We can use matrix algebra to solve for the currents I_1 and I_2 in terms of V_1 and V_2 ; the answer will of course be identical to that given above.

More generally, if there are n independent loops in the circuit, Kirchhoff's laws provide n equations that can be written in the form

$$\tilde{V} = \mathbf{R} \cdot \tilde{I} ,$$

where

$$\tilde{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} , \quad \tilde{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} ,$$

and where \mathbf{R} is an $n \times n$ matrix whose elements all have the units of resistance. This may sound hideous ... *but*, the solution is now relatively simple: we "simply" invert the matrix, and find the n currents from the equation

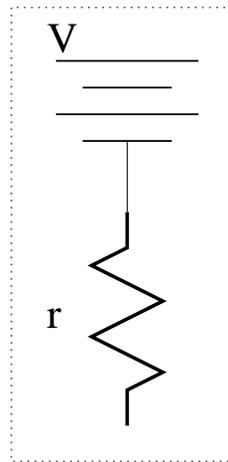
$$\tilde{I} = \mathbf{R}^{-1} \cdot \tilde{V} .$$

This is the only practical way to solve for the currents in circuits that have lots of loops. Doing the inversion isn't even all that bad, thanks to computerized math tools. For very large matrices ($n > 100$ or so), very fast matrix inverters exist for numerically specified matrices.

8.6 EMF and batteries revisited: Internal resistance

One final issue of practice is often of great importance when working with batteries. When a battery is delivering current to a circuit, there is necessarily a flow of current *within the battery itself*. In the example discussed at the start of this lecture, this current comes from the motion of H^+ ions from the Pb to the PbO_2 .

Because this reaction proceeds chemically, these reactions will dissipate energy; the battery gets hot, and energy which could have gone into the circuit is lost. This is described mathematically by giving the battery an *internal resistance* r . For the purpose of any calculation, the internal resistance is taken into account by simply putting a small resistor r in series with the battery. This is usually drawn something like this:



An important consequence of internal resistance is that a battery has a maximum current it can deliver: $I_{\max} = V/r$. Roughly speaking, its chemical reactions cannot proceed fast enough to deliver more than this current. When this happens, a power $I_{\max}^2 r$ is dissipated in the battery, which gets hot in a hurry. A good way to prove to yourself the reality of internal resistance is to carry a nine-volt battery in a pocket full of keys and change. The pieces of metal short out the battery's terminals, it gets hot, you scream in pain and hop around like a lunatic².

²Yes, I speak from personal experience.