*Scott Hughes* *10 March 2005*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.022 SPRING 2004

LECTURE 10:
MAGNETIC FORCE; MAGNETIC FIELDS; AMPERE'S LAW

## 10.1 The Lorentz force law

Until now, we have been concerned with electrostatics — the forces generated by and acting upon charges at rest. We now begin to consider how things change when charges are in motion[1].

A simple apparatus demonstrates that something wierd happens when charges are in motion: If we run currents next to one another in parallel, we find that they are *attracted* when the currents run in the same direction; they are *repulsed* when the currents run in opposite directions. This is despite the fact the wires are *completely neutral*: if we put a stationary test charge near the wires, it feels no force.

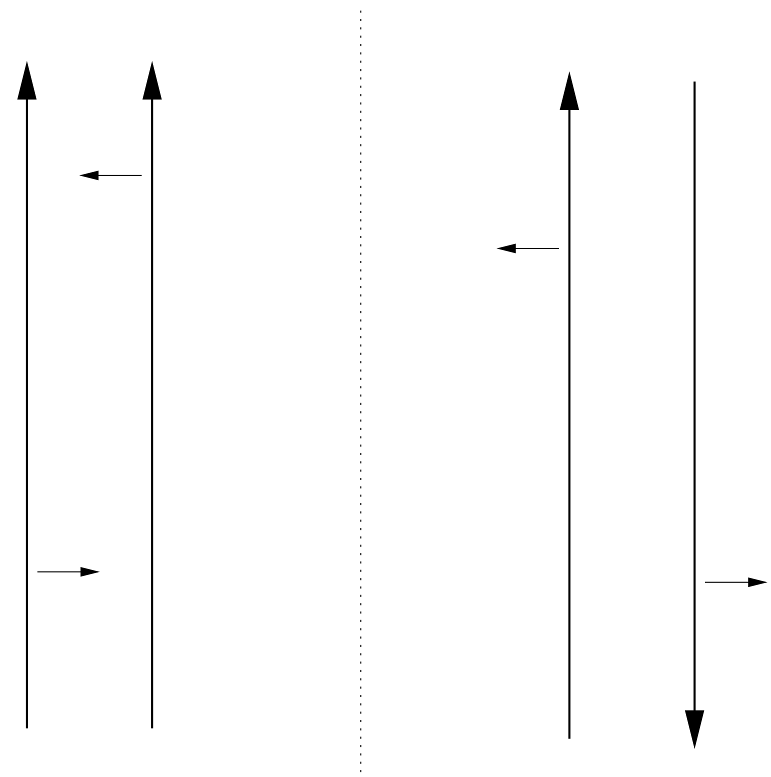


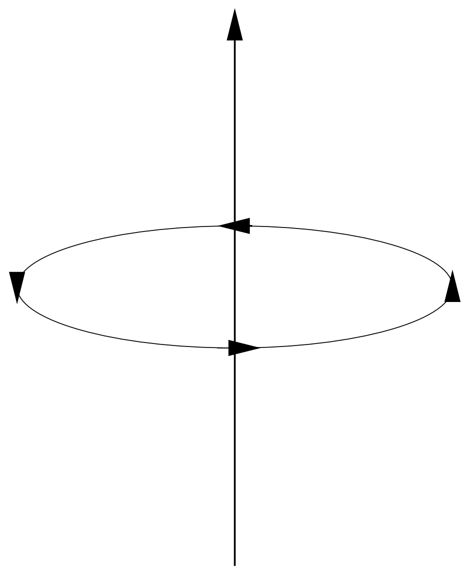
Figure 1: Left: parallel currents attract. Right: Anti-parallel currents repel.

Furthermore, experiments show that the force is proportional to the currents — double the current in *one* of the wires, and you double the force. Double the current in both wires, and you quadruple the force.

[1] We will deviate a bit from Purcell's approach at this point. In particular, we will defer our discussion of special relativity til next lecture.

This all indicates a force that is proportional to the velocity of a moving charge; and, that points in a direction *perpendicular* to the velocity. These conditions are screaming for a force that depends on a cross product.

What we say is that some kind of field $\vec{B}$ — the "magnetic field" — arises from the current. (We'll talk about this in detail very soon; for the time being, just accept this.) The direction of this field is kind of odd: it wraps around the current in a circular fashion, with a direction that is defined by the right-hand rule: We point our right thumb in the direction of the current, and our fingers curl in the same sense as the magnetic field.



With this sense of the magnetic field defined, the force that arises when a charge moves through this field is given by

$$\vec{F} = q\frac{\vec{v}}{c} \times \vec{B} \ ,$$

where $c$ is the speed of light. The appearance of $c$ in this force law is a hint that special relativity plays an important role in these discussions.

If we have both electric and magnetic fields, the *total* force that acts on a charge is of course given by

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \ .$$

This combined force law is known as the Lorentz force.

### 10.1.1 Units

The magnetic force law we've given is of course in cgs units, in keeping with Purcell's system. The magnetic force equation itself takes a slightly different form in SI units: we do not include the factor of $1/c$, instead writing the force

$$\vec{F} = q\vec{v} \times \vec{B} \ .$$

This is a very important difference! It makes comparing magnetic effects between SI and cgs units slightly nasty.

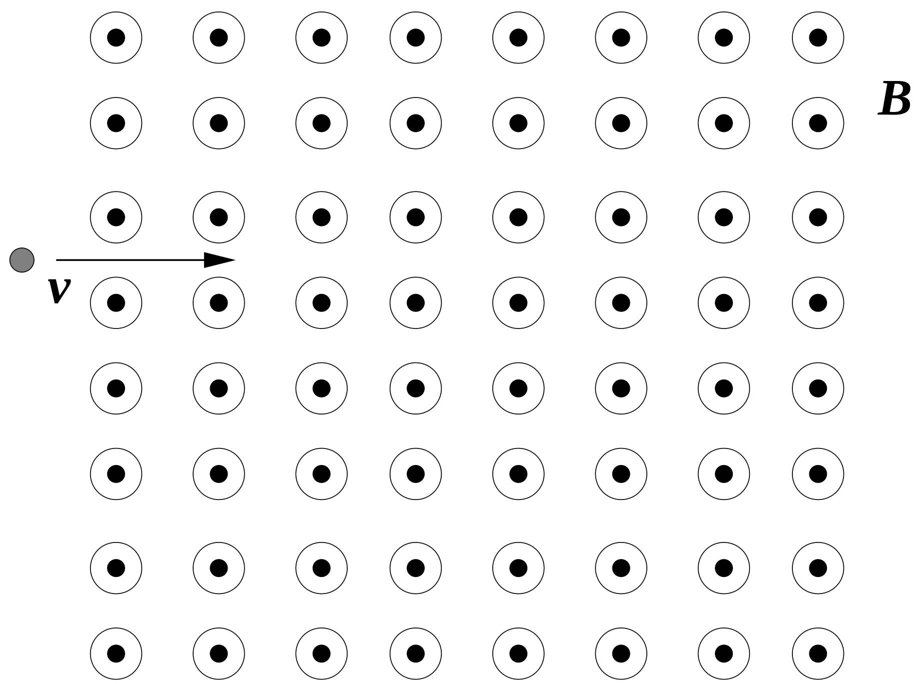
Notice that, in cgs units, the magnetic field has the same overall dimension as the electric field: $\vec{v}$ and $c$ are in the same units, so $\vec{B}$ must be force/charge. For historical reasons, this combination is given a special name: 1 dyne/esu equals 1 Gauss (1 G) when the force in question is magnetic. (There is no special name for this combination when the force is electric.)

In SI units, the magnetic field does *not* have the same dimension as the electric field: $\vec{B}$ must be force/(velocity $\times$ charge). The SI unit of magnetic field is called the Tesla (T): the Tesla equals a Newton/(coulomb $\times$ meter/sec).

To convert: $1\,\mathrm{T} = 10^4\,\mathrm{G}$.

## 10.2 Consequences of magnetic force

Suppose I shoot a charge into a region filled with a uniform magnetic field:




The magnetic field $\vec{B}$ points out of the page; the velocity $\vec{v}$ initially points to the right. What motion results from the magnetic force?

At every instant, the magnetic force points perpendicular to the charge's velocity — exactly the force needed to cause circular motion. It is easy to find the radius of this motion: if the particle has charge $q$ and mass $m$, then

$$\begin{aligned} F_{\mathrm{mag}} &= F_{\mathrm{centripetal}} \\ \frac{qvB}{c} &= \frac{mv^2}{R} \\ R &= \frac{mvc}{qB} \,. \end{aligned}$$

If the charge $q$ is positive, the particle's trajectory veers to the right, vice versa if its negative. This kind of qualitative behavior — bending the motion of charges along a curve — is typical of magnetic forces.

Notice that magnetic forces do no work on moving charges: if we imagine the charge moves for a time $dt$, the work that is done is

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{s} = \vec{F} \cdot \vec{v}\, dt \\ &= q\left(\frac{\vec{v}}{c} \times \vec{B}\right) \cdot \vec{v}\, dt \\ &= 0\,. \end{aligned}$$

The zero follows from the fact that $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$.

## 10.3 Force on a current

Since a current consists of a stream of freely moving charges, a magnetic field will exert a force upon any flowing current. We can work out this force from the general magnetic force law.

Consider a current $I$ that flows down a wire. This current consists of some linear density of freely flowing charges, $\lambda$, moving with velocity $\vec{v}$. (The direction of the charges' motion is defined by the wire: they are constrained by the wire's geometry to flow in the direction it points.) Look at a little differential length $dl$ of this wire (a vector, since the wire defines the direction of current flow).

The amount of charge contained in this differential length is $dq = \lambda\, dl$. The differential of force exerted on this piece of the wire is then

$$d\vec{F} = (\lambda\, dl)\frac{\vec{v}}{c} \times \vec{B}\,.$$

There are two equivalent ways to rewrite this in terms of the current. First, because the current is effectively a vector by virtue of the velocity of its constituent charges, we put $\vec{I} = \lambda\vec{v}$ and find

$$d\vec{F} = dl\,\frac{\vec{I}}{c} \times \vec{B}\,.$$

Second, we can take the current to be a scalar, and use the geometry of the wire to define the vector:

$$d\vec{F} = \frac{I}{c} d\vec{l} \times \vec{B}\,.$$

These two formulas are completely equivalent to one another. Let's focus on the second version. The total force is given by integrating:

$$\vec{F} = \frac{I}{c}\int d\vec{l} \times \vec{B}\,.$$

If we have a long, straight wire whose length is $L$ and is oriented in the $\hat{n}$ direction, we find

$$\vec{F} = \frac{IL}{c}\hat{n} \times \vec{B}\,.$$

This formula is often written in terms of the force per unit length: $\vec{F}/L = (I/c)\hat{n} \times \vec{B}$.

## 10.4 Ampere's law

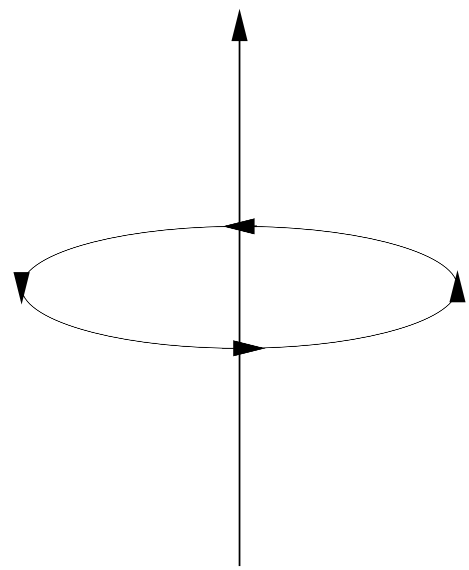
We've talked about the force that a magnetic field exerts on charges and current; but, we have not yet said anything about where this field comes from. I will now give, without *any* proof or motivation, a few key results that allow us to determine the magnetic field in many situations.

The main result we need is *Ampere's law*:

$$\oint_C \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} I_{\rm encl} \ .$$

In words, if we take the line integral of the magnetic field around a closed path, it equals $4\pi/c$ times the current enclosed by the path.

Ampere's law plays a role for magnetic fields that is similar to that played by Gauss's law for electric fields. In particular, we can use it to calculate the magnetic field in situations that are sufficiently symmetric. An important example is the magnetic field of a long, straight wire: In this situation, the magnetic field must be constant on any circular path around the



wire. The amount of current enclosed by this path is just $I$, the current flowing in the wire:

$$\oint \vec{B} \cdot d\vec{s} = B(r) 2\pi r = \frac{4\pi}{c} I$$

$$\rightarrow B(r) = \frac{2I}{cr} \ .$$

The magnetic field from a current thus falls off as $1/r$. Recall that we saw a similar $1/r$ law not so long ago — the electric field of a long line charge also falls off as $1/r$. As we'll see fairly soon, this is not a coincidence.

The direction of this field is in a "circulational" sense — the $\vec{B}$ field winds around the wire according to the right-hand rule[2]. This direction is often written $\hat{\phi}$, the direction of

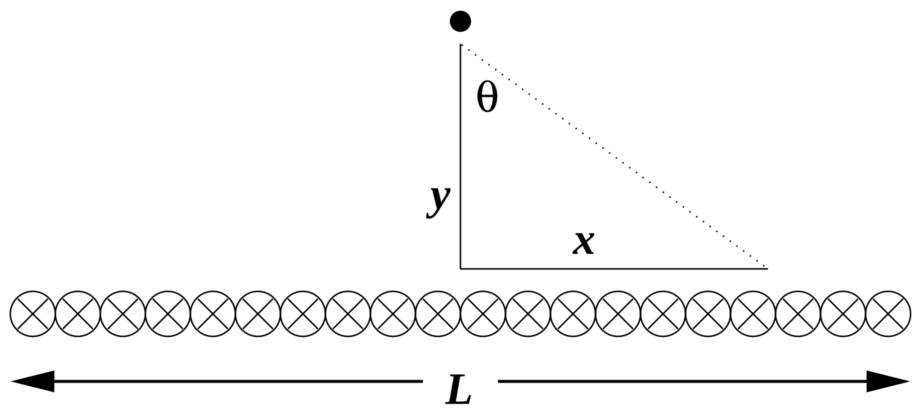
[2]In principle, we could have defined it using a "left-hand rule". This would give a fully consistent description of physics provided we switched the order of all cross products (which is identical to switching the sign of all cross products).

increasing polar angle $\phi$. The full vector magnetic field is thus written

$$\vec{B} = \frac{2I}{cr}\hat{\phi} \; .$$

### 10.4.1 Field of a plane of current

The magnetic field of the long wire can be used to derive one more important result. Suppose we take a whole bunch of wires and lay them next to each other:




The current in each wire is taken to go into the page. Suppose that the total amount of current flowing in *all* of the wires is $I$, so that the current *per unit length* is $K = I/L$. What is the magnetic field at a distance $y$ above the center of the plane?

This is fairly simple to work out using superposition. First, from the symmetry, you should be able to see that only the horizontal component of the magnetic field (pointing to the right) will survive. For the vertical components, there will be equal and opposite contributions from wires left and right of the center. To sum what's left, we set up an integral:

$$\vec{B} = \frac{2}{c}\hat{x}\int_{-L/2}^{L/2}\frac{(I/L)dx}{\sqrt{x^2+y^2}}\cos\theta \; .$$

In the numerator under the integral, we are using $(I/L)dx$ as the current carried by a "wire" of width $dx$. Doing the trigonometry, we replace $\cos\theta$ with something a little more useful:

$$\begin{aligned}
\vec{B} &= \frac{2}{c}\hat{x}\int_{-L/2}^{L/2}\frac{(I/L)dx}{\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}} \\
&= \frac{2Ky}{c}\hat{x}\int_{-L/2}^{L/2}\frac{dx}{x^2+y^2} \; .
\end{aligned}$$

This integral is doable, but not particularly pretty (you end up with a mess involving arctangents). A more tractable form is obtaining by taking the limit of $L \to \infty$: using

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+y^2} = \frac{\pi}{|y|} \; ,$$

we find

$$\vec{B} = \hat{x}\frac{2\pi Ky}{c|y|}$$

$$
\begin{aligned}
&= +\hat{x}\frac{2\pi K}{c} \qquad y > 0 \quad \text{(above the plane)} \\
&= -\hat{x}\frac{2\pi K}{c} \qquad y < 0 \quad \text{(below the plane)}
\end{aligned}
$$

The most important thing to note here is the change as we cross the sheet of current:

$$|\Delta\vec{B}| = \frac{4\pi K}{c}\ .$$

Does this remind you of anything? It should! When we cross a sheet of *charge* we have

$$|\Delta\vec{E}| = 4\pi\sigma\ .$$

The sheet of current plays a role in magnetic fields very similar to that played by the sheet of charge for electric fields.

### 10.4.2 Force between two wires

Combining the result for the magnetic field from a wire with current $I_1$ with the force per unit length upon a long wire with current $I_2$ tells us the force per unit length that arises between two wires:

$$\frac{|\vec{F}|}{L} = \frac{2I_1I_2}{c^2 r}\ .$$

Using right-hand rule, you should be able to convince yourself quite easily that this force is attractive when the currents flow in the same direction, and is repulsive when they flow in opposite directions.

### 10.4.3 SI units

In SI units, Ampere's law takes the form

$$\oint_C \vec{B}\cdot d\vec{S} = \mu_0 I_{\text{encl}}$$

where the constant $\mu_0 = 4\pi \times 10^{-7}$ Newtons/amp$^2$ is called the "magnetic permeability of free space". To convert any cgs formula for magnetic field to SI, multiply by $\mu_0 \times (c/4\pi)$. For example, the magnetic field of a wire becomes

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}\ .$$

The force between two wires becomes

$$\frac{|\vec{F}|}{L} = \frac{\mu_0 I_1 I_2}{2\pi r}\ .$$

If you try to reproduce this force formula, remember that the magnetic force in SI units does not have the factor $1/c$.

## 10.5 Divergence of the $\vec{B}$ field

Let's take the divergence of straight wire's magnetic field using Cartesian coordinates. We put $r = \sqrt{x^2 + y^2}$. With a little trigonometry, you should be able to convince yourself that

$$\begin{aligned}
\hat{\phi} &= \hat{y}\cos\phi - \hat{x}\sin\phi \\
&= \frac{x\hat{y}}{\sqrt{x^2+y^2}} - \frac{y\hat{x}}{\sqrt{x^2+y^2}} \,,
\end{aligned}$$

so

$$\vec{B} = \frac{2I}{c}\left[\frac{x\hat{y}}{x^2+y^2} - \frac{y\hat{x}}{x^2+y^2}\right] .$$

The divergence of this field is

$$\begin{aligned}
\vec{\nabla}\cdot\vec{B} &= \frac{2I}{c}\left[\frac{2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right] \\
&= 0 \,.
\end{aligned}$$

We could have guessed this without doing any calculation: if we make any small box, there will be just as many field lines entering it as leaving.

Although we have only done this in detail for this very special case, it turns out this result holds for magnetic fields in general:

$$\boxed{\boxed{\vec{\nabla}\cdot\vec{B} = 0}}$$

Recall that we found $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ — the divergence of the electric field told us about the density of electric charge. The result $\vec{\nabla}\cdot\vec{B} = 0$ thus tells us that there is *no such thing as magnetic charge*.

This is actually not a foregone conclusion: there are reasons to believe that very small amounts of magnetic charge may exist in the universe, created by processes in the big bang. If any such charge exists, it would create a "Coulomb-like" magnetic field, with a form just like the electric field of a point charge. No conclusive evidence for these monopoles has ever been found; but, absence of evidence is not evidence of absence.

## 10.6 What *is* the magnetic field???

This "magnetic field" is, so far, just a construct that may seem like I've pulled out of the air. I haven't pulled it out of the air just for kicks — observations and measurements demonstrate that there is an additional field that only acts on moving charges. But what exactly is this field? Why should there exist some field that only acts on *moving* charges?

The answer is to be found in special relativity. The defining postulate of special relativity essentially tells us that physics must be *consistent* in every "frame of reference". Frames of reference are defined by observers moving, with respect to each other, at different velocities.

Consider, for example, a long wire in some laboratory that carries a current $I$. In this "lab frame", the wire generates a magnetic field. Suppose that a charge moves with velocity $\vec{v}$ parallel to this wire. The magnetic field of the wire leads to an attractive force between the charge and the wire.

Suppose we now examine this situation from the point of view of the charge (the "charge frame"). From the charge's point of view, it is sitting perfectly still. If it is sitting still, **there can be no magnetic force!** We appear to have a problem: in the "lab frame", there is an attractive magnetic force. In the "charge frame", there can't possibly be an attractive magnetic force. But for physics to be consistent in both frames of reference, there must be *some* attractive force in the charge frame. What is it???

There's only thing it can be: in the charge's frame of reference, **there must be an attractive ELECTRIC field.** In other words, what looks like a pure magnetic field in one frame of reference looks (at least in part) like an electric field in another frame of reference. To understand how this happens, we must begin to understand special relativity. This is our next topic.