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LECTURE 13:
AMPERE'S LAW REVISITED; BIOT-SAVART LAW
VECTOR POTENTIAL

13.1 Return to the magnetic field

13.1.1 What we did before

Let's quickly recap the major facts we've gone over regarding the magnetic field. A charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ feels a force

$$\vec{F} = q\frac{\vec{v}}{c} \times \vec{B} \; .$$

Magnetic fields arise from flowing currents. The integral of $\vec{B}$ along a closed path C the is related to the current enclosed by the path by Ampere's law:

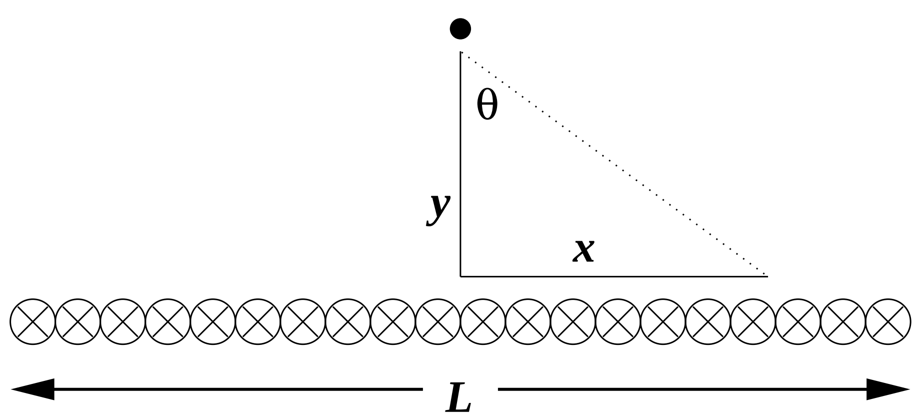
$$\oint_C \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} I_{\rm encl} \; .$$

If we apply this law to a long wire we find

$$\vec{B}(r) = \frac{2I}{cr}\hat{\phi} \; .$$

where r is the distance to the wire and $\hat{\phi}$ is related to the direction in which the current flows by the right-hand rule: point your right-hand thumb in the direction in which the current flows and $\hat{\phi}$ curls around in the direction of your fingers.

By placing many long wires next to each other and invoking superposition, we found the magnetic field of a sheet of current:



In the limit of $L \to \infty$, but holding the current per unit length in the sheet $K \equiv I/L$ constant, we found that

$$\vec{B} = +\frac{2\pi K}{c}\hat{x}$$

above the plane, and

$$\vec{B} = -\frac{2\pi K}{c}\hat{x}$$

below the plane. The key observation here is that the *change* in the magnitude of the magnetic field as we cross the current is given by

$$|\Delta\vec{B}| = \frac{4\pi K}{c}\ .$$

This should be reminiscent of the change in electric field when we cross a sheet of charge, $|\Delta\vec{E}| = 4\pi\sigma$.

Taking the divergence of the wire's $\vec{B}$-field, we find

$$\vec{\nabla}\cdot\vec{B} = 0\ .$$

Although we have only shown this holds for a specific example, this rule holds for all magnetic fields that arise from currents. Physically, it tells us that — as far we can tell — there is no such thing as an isolated magnetic charge. The smallest "chunks" of magnetic field that we can find come in a dipole configuration, with "north" and "south" poles. We never find an isolated "north" pole, with radial field lines.

13.1.2 Ampere's law revisited

Using Stoke's theorem, we can rewrite the line integral of the $\vec{B}$ field in terms of its curl:

$$\oint_C \vec{B}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a}\ .$$

S is any surface bounded by the curve C. This is a nice result, since the current enclosed by C is given by an integral through the *same* surface S:

$$I_{\rm encl} = \int_S \vec{J}\cdot d\vec{a}\ .$$

Putting these two results together, Ampere's law becomes

$$\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \frac{4\pi}{c}\int_S \vec{J}\cdot d\vec{a}$$
$$\int_S \left(\vec{\nabla}\times\vec{B} - \frac{4\pi}{c}\vec{J}\right)\cdot d\vec{a} = 0\ .$$

We finally obtain

$$\longrightarrow \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}\ .$$

This last equation is also called Ampere's law — it expresses the same physics as $\oint_C \vec{B}\cdot d\vec{s} = 4\pi I/c$, but in differential rather than integral form. If we had been doing this analysis in SI units rather than cgs, we would have ended up with

$$\longrightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J}\ .$$

It's worthwhile to pause at this moment and collect the various divergence and curl equations for $\vec{E}$ and $\vec{B}$ that we have so far:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho\,, \qquad \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} &= 0\,, \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}\,. \end{aligned}$$

These are Maxwell's equations for static fields and steady currents. They are what you see on the t-shirts whenever all time derivatives go to zero. Their SI versions are

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_0\,, \qquad \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} &= 0\,, \qquad \vec{\nabla} \times \vec{B} = \mu_0\vec{J}\,. \end{aligned}$$

13.2 The vector potential

13.2.1 Definition

Recall that for static electric fields, we have

$$\vec{E} = -\vec{\nabla}\phi\,.$$

We motivated this rule from arguments based on work and energy — namely, that the potential difference between any two points is just the work it takes (per unit charge) to move a charge between those two points. *However*, it has an important and useful side effect: it "guarantees" that our electric fields are curl free:

$$\vec{\nabla} \times \vec{E} = 0$$

since

$$\vec{\nabla} \times \vec{\nabla} f = 0$$

for any scalar function f.

Can we do anything similar with the magnetic field? In this case, the equation which we would like to enforce is the rule

$$\vec{\nabla} \cdot \vec{B} = 0\,.$$

Can we pick a form of $\vec{B}$ that guarantees this equation holds? Well, obviously, or I wouldn't have wasted the time to type this up: as you proved on Pset 3, the divergence of the curl of any function is zero. This suggests that we can write $\vec{B}$ as the curl of some vector function $\vec{A}$:

$$\vec{\nabla} \times \vec{A} = \vec{B}\,.$$

The function $\vec{A}$ is called the "vector potential". The key thing to bear in mind is that it is introduced to enforce the rule $\vec{\nabla} \cdot \vec{B} = 0$, in the same way that $\vec{E} = -\vec{\nabla}\phi$ enforces $\vec{\nabla} \times \vec{E} = 0$. Unlike the scalar potential ϕ, however, $\vec{A}$ has *no connection* to work or energy. For our purposes at least, its only use is to facilitate some of our calculations[1].

[1] It turns out to have a very real physical meaning in quantum mechanics, though.

13.2.2 Non-uniqueness

In electrostatics, we were able to prove a uniqueness theorem: once boundary conditions were set, there was one — and *only* one — potential for a given charge distribution, meaning that for every $\vec{E}$ field there was only one potential ϕ.

This is *not* the case for the vector potential! It is actually the case that an *infinite* number of vector potential functions correspond to a particular magnetic field $\vec{B}$. Consider a uniform magnetic field: $\vec{B} = B_0\hat{z}$. Our only requirement on the vector potential $\vec{A}$ is that its curl reproduce this field. In other words, we need

$$\begin{aligned}
B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0 \\
B_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \\
B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0 \ .
\end{aligned}$$

One vector potential which satisfies these conditions is

$$\vec{A} = xB_0\hat{y} \ .$$

Another is

$$\vec{A} = -yB_0\hat{x} \ .$$

Still another is given by combining these:

$$\vec{A} = -\frac{1}{2}yB_0\hat{x} + \frac{1}{2}xB_0\hat{y} \ .$$

These are *all* valid choices. Given this freedom, it might seem like there's no point in selecting any particular solution. In fact, we end up taking the opposite viewpoint: Since any one of these solutions is valid, let's pick the one that is — in some manner that we must specify — particularly convenient. In the next subsection, we describe a particular choice that allows us to write down an equation that looks a lot like Poisson's equation, but that holds for $\vec{A}$.

13.2.3 Poisson's equation for $\vec{A}$

When we plugged the definition $\vec{E} = -\vec{\nabla}\phi$ into Gauss's law, $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, we ended up with Poisson's equation,

$$\nabla^2\phi = -4\pi\rho \ .$$

What happens if we plug $\vec{\nabla}\times\vec{A} = \vec{B}$ into Ampere's law, $\vec{\nabla}\times\vec{B} = 4\pi\vec{J}/c$? First, we need to work out the curl of the curl: with a fair amount of tedious algebra you should be able to to show that

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A} \ .$$

This is in principle a horrible mess. However, we can take advantage of the non-uniqueness of $\vec{A}$ to simplify it. We have the freedom to *choose* our $\vec{A}$ in such a way that it has no divergence:

$$\vec{\nabla}\cdot\vec{A} = 0 \ .$$

This is called a "gauge choice". Some of you will study this notion in *far* greater detail in future courses. For now, you simply need to take away the notion that the non-uniqueness of $\vec{A}$ gives us enough freedom to tinker around until the condition $\vec{\nabla}\cdot\vec{A}=0$ is guaranteed to be met. (It still leaves us a lot of freedom. You can easily verify that all three of the choices for $\vec{A}$ we wrote down in the previous subsection satisfy $\vec{\nabla}\cdot\vec{A}=0$.)

Imposing this condition, we then have

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = -\nabla^2\vec{A}\,.$$

Let's now plug this into Ampere's law: we have

$$\begin{aligned}
\vec{\nabla}\times\vec{B} &= \frac{4\pi}{c}\vec{J} \\
\vec{\nabla}\times\vec{\nabla}\times\vec{A} &= \frac{4\pi}{c}\vec{J} \\
\longrightarrow \nabla^2\vec{A} &= -\frac{4\pi}{c}\vec{J}\,.
\end{aligned}$$

This is kind of a goofy equation. In cartesian coordinates, it is best understood component by component. Basically, it means that each component of $\vec{A}$ is related to a component of $\vec{J}$ by the Laplacian operator:

$$\begin{aligned}
\nabla^2 A_x &= -\frac{4\pi}{c}J_x \\
\nabla^2 A_y &= -\frac{4\pi}{c}J_y \\
\nabla^2 A_z &= -\frac{4\pi}{c}J_z\,.
\end{aligned}$$

Let's look back at electrostatics again for a moment. Poisson's equation, $\nabla^2\phi=-4\pi\rho$, has as its solution the general equation for potential,

$$\phi = \int_V \frac{\rho\,dV}{r}\,.$$

Our Poisson equation for the vector potential is the *exact same thing*, provided we replace ϕ with $\vec{A}$ and ρ with $\vec{J}/c$:

$$\vec{A} = \frac{1}{c}\int_V \frac{\vec{J}\,dV}{r}\,.$$

For a current I flowing in a wire, this equation becomes

$$\vec{A} = \frac{I}{c}\int_{\text{along wire}} \frac{d\vec{l}}{r}\,,$$

where $d\vec{l}$ points along the wire.

13.3 The Biot-Savart law

Using the last result we derived — the vector potential of wire — we can work out another very important result. By taking the curl of that vector potential, we derive the magnetic field of a wire:

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \frac{I}{c} \int_{\text{along wire}} \frac{d\vec{l}}{r} \\ &= \frac{I}{c} \int_{\text{along wire}} \vec{\nabla} \times \frac{d\vec{l}}{r} \,.\end{aligned}$$

We can take the "$\vec{\nabla}\times$" under the integral since the limits of integration do not change when we take the derivative. Next, we use

$$\vec{\nabla} \times \frac{d\vec{l}}{r} = \left(\frac{1}{r}\right) \vec{\nabla} \times d\vec{l} + \vec{\nabla}\left(\frac{1}{r}\right) \times d\vec{l} \,.$$

(This follows from stuff you did on pset 3.) $\vec{\nabla} \times d\vec{l} = 0$ since $d\vec{l}$ is a just a differential length along the wire, and $\vec{\nabla}(1/r) = -\hat{r}/r^2$. We find

$$\vec{\nabla} \times \frac{d\vec{l}}{r} = d\vec{l} \times \frac{\hat{r}}{r^2}$$

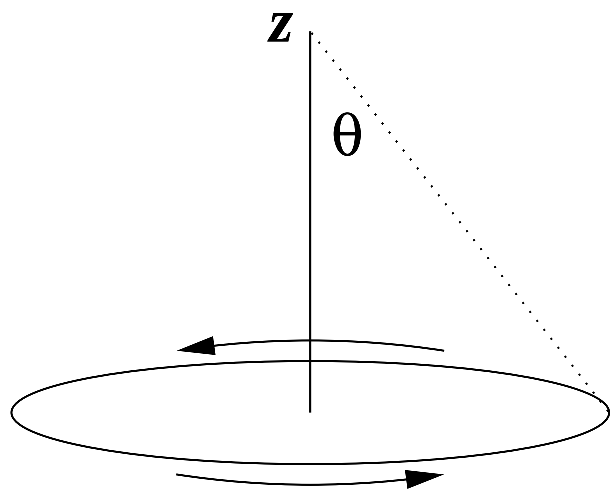
where we've used $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ to get rid of the minus sign. We thus finally can write down the magnetic field of the wire:

$$\vec{B} = \frac{I}{c} \int_{\text{along wire}} \frac{d\vec{l} \times \hat{r}}{r^2} \,.$$

This result is known as the *Biot-Savart* formula for the magnetic field. It plays roughly the same role in determining the magnetic field from a current that Coulomb's law plays setting the electric field from charges.

13.3.1 Field of a ring of current

The Biot-Savart law allows us to work out the magnetic field of an interesting special case: a ring of current.



Take the ring to have radius R, and consider a current of magnitude I circulating as shown. What is the magnitude and direction of the current a point z above the ring's center?

Set up Biot-Savart:

$$\vec{B} = \frac{I}{c}\int_{\text{around ring}} \frac{d\vec{l} \times \hat{r}}{r^2}\ .$$

The distance r is easy — $r = \sqrt{z^2 + R^2}$. Likewise, the magnitude $|d\vec{l}|$ is easy — $|d\vec{l}| = R d\phi$, where ϕ is an angle around the perimeter of the ring. The cross product $d\vec{l} \times \hat{r}$ is a little bit trickier. A good way to understand it is to consider a special case: when $z = 0$, $d\vec{l} \times \hat{r}$ points in the vertical $\hat{z}$ direction. Moving away from this case, we see that it *always* points along $\hat{z}$, but is reduced by a factor of $\sin\theta$ in general:

$$\begin{aligned} \vec{B} &= \frac{I}{c}\int_0^{2\pi} \frac{R\sin\theta\, d\phi}{R^2 + z^2}\hat{z} \\ &= \frac{I}{c}\int_0^{2\pi} \frac{R^2\, d\phi}{(R^2 + z^2)^{3/2}}\hat{z} \\ &= \frac{2\pi I}{c}\frac{R^2}{(R^2 + z^2)^{3/2}}\hat{z}\ . \end{aligned}$$

Notice that the direction, $\hat{z}$, could have been predicted by the right-hand rule: curl your fingers of your right hand in the direction of the current, and your thumb points along the field. Note also that it points in the same direction above and below the loop.

13.3.2 The solenoid

Suppose we stack a bunch of rings like this on top of one another. What is the magnetic field at the center of the cylinder that we make?

We answer this by adding up the magnetic fields from many rings. Let the total height of the stack be L. Further, let the total amount of current circulating through *all* of the rings be I. The total current per unit length flowing through the stack is thus $K = I/L$. The contribution to the magnetic field from a height z above the center and from a ring of thickness dz is then

$$\begin{aligned} d\vec{B} &= \frac{2\pi}{c}\frac{I\,dz}{L}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{z} \\ &= \frac{2K\pi}{c}\frac{R^2\,dz}{(R^2+z^2)^{3/2}}\hat{z} \end{aligned}$$

The *total* magnetic field is given by integrating this over z:

$$\vec{B} = \frac{2\pi K}{c}\hat{z}\int_{-L/2}^{L/2}\frac{R^2\,dz}{(R^2+z^2)^{3/2}} \ .$$

This integral turns out to be quite easy to do: since

$$\int_{-L/2}^{L/2}\frac{R^2\,dz}{(R^2+z^2)^{3/2}} = \frac{2L}{\sqrt{L^2+4R^2}}$$

the answer is just

$$\vec{B} = \frac{4\pi K}{c}\frac{L}{\sqrt{L^2+4R^2}}\hat{z} \quad \to \frac{4\pi K\hat{z}}{c} \qquad \text{as } L\to\infty \ .$$

This configuration is known as a “solenoid”. The last formula (the $L \to \infty$ limit) is accurate enough (provided $L \gg R$) that it is often used even for finite length solenoids.

This value turns out to describe the magnetic field *throughout* the interior for $L \gg R$! Note that the value is exactly what we got for change of field when we cross a current sheet: in a very long solenoid, there is no field on the outside, but it suddenly jumps to $4\pi K/c$ on the inside. Because of this, solenoids play a role with for magnetic fields similar that played by parallel plate capacitors for electric fields: they are devices which “contain” the field in some isolated region.

In practice, solenoids are constructed by wrapping wires around a tube to make a cylindrical geometry. If we can wrap our wires n times per unit length (for example, we may have $n = 10$ turns/cm), then $K = nI$.