

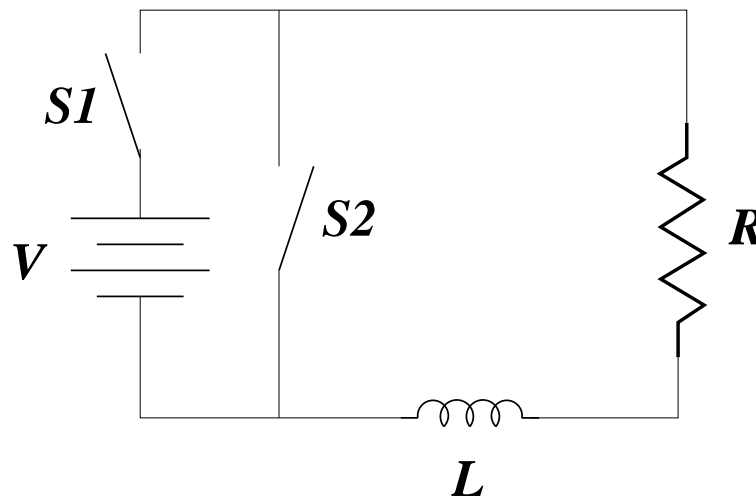
MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
 DEPARTMENT OF PHYSICS  
 8.022 SPRING 2005

LECTURE 16:  
 RL CIRCUITS.

UNDRIVEN RLC CIRCUITS; PHASOR REPRESENTATION.

## 16.1 RL circuits

Now that we know all about inductance, it's time to start thinking about how to use it in circuits. A simple circuit that illustrates the major concepts of inductive circuits is this:



Suppose that at  $t = 0$ , we close the switch  $S1$ , leaving  $S2$  open. How does the current evolve in the circuit after this?

Let's first think about this physically. Lenz's law tells us that the magnetic flux through the inductor does not "want" to change. Since this flux is initially zero, the inductor will impede any current that tries to flow it — the current will initially try to remain at zero. As time passes, current will gradually leak through, and the magnetic flux will build up. Eventually, it should saturate at  $I = V/R$  — at this point the most current possible is flowing through the circuit.

### 16.1.1 To Kirchhoff or not to Kirchhoff

We'd now like to analyze this circuit quantitatively as well. In all other circumstances in which we've examined a circuit, we've used Kirchhoff's laws for this; for a simple single-loop circuit, the relevant rule is Kirchhoff's second law, "The sum of the EMFs and the voltage drops around a closed loops is zero".

This rule is essentially a restatement of the old electrostatics rule that  $\oint \vec{E} \cdot d\vec{s} = 0$ . **This rule DOES NOT HOLD when we have a time changing magnetic field!** In other words, inductors completely invalidate — at least formally — the foundation of Kirchhoff's second law.

So what do we do? Let's reconsider: in place of  $\oint \vec{E} \cdot d\vec{s} = 0$ , we have Faraday's law:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{1}{c} \frac{d\Phi_B}{dt} = \mathcal{E}_{\text{ind}} = L \frac{dI}{dt} .$$

The closed loop integral of  $\vec{E}$  gives us the induced EMF. I personally find it best to think of the inductor as an EMF source; it acts in the circuit effectively like a battery would. Even though Kirchhoff's laws no longer stand, formally, on a very solid foundation, we can extend the formalism by thinking of inductors in the circuit as a source of EMF. The equations then carry over just fine; indeed, they carry over so well that almost all textbooks tell you to just apply Kirchhoff's laws to circuits containing inductors without even mentioning this subtlety. It's always worth bearing in mind that what you are *really* using is Faraday's law. I will try to refer to the loop rule as Faraday's law; it's such a habit to call it Kirchhoff's law, though, that I will surely slip from time to time.

The tricky thing in these kinds of problems is getting the signs right. In *all* of these cases, Lenz's law should guide you: the induced EMF will act in such a way as to oppose changes in the inductor's magnetic flux. If a circuit initially has no current flowing through it, the inductor will oppose the build-up of current; if it initially *does* have current flowing, the inductor will try to keep that current flowing. Keep this intuition in mind and you should have *no* problem choosing the correct sign for the inductor's EMF.

(Very detailed discussion of inductors and the failure of Kirchhoff's laws can be found in Walter Lewin's OCW 8.02 lectures from 2002; see Lectures 15 and 20 in particular.)

### 16.1.2 Back to our regularly scheduled program

OK, let's calculate. The EMF due to the battery is  $+V$ ; the EMF induced in the inductor is  $-LdI/dt$ . Why the minus sign? It's going to fight the growth in current, and so must be opposite to the battery. Finally, there's a voltage drop  $-IR$  at the resistor:

$$V - IR - L \frac{dI}{dt} = 0 .$$

Rearrange:

$$\begin{aligned} \frac{dI}{dt} &= \frac{V - IR}{L} \\ &= \frac{R}{L} \left( \frac{V}{R} - I \right) . \end{aligned}$$

Now divide; we end up with something that is of the form  $du/u$ :

$$\frac{dI}{I - V/R} = -\frac{R dt}{L}$$

so it is easy to integrate up, using the boundary conditions  $I = 0$  at  $t = 0$ ,  $I = I(t)$  at some general time  $t$ :

$$\ln \left[ \frac{I(t) - V/R}{-V/R} \right] = -\frac{Rt}{L}$$

or

$$I(t) = \frac{V}{R} \left( 1 - e^{-Rt/L} \right) .$$

As we guessed on physical grounds, the current builds up, with a “growth timescale”  $L/R$ . The timescale  $L/R$  plays a role in LR circuits similar to that played by the timescale  $RC$  in circuits with capacitors.

Suppose we let this circuit sit in this state for a very long time, so that the current saturates at its equilibrium value,  $I = V/R$ . What happens if we now simultaneously open  $S1$  and close  $S2$ ? Physically, our expectation is that Lenz’s law will try to hold the current flow constant — the inductor says “I don’t want the flux through me to change!” However, we know that the resistor dissipates energy, so the current must bleed away — we should see some kind of exponential decay.

Time to calculate. Going around the circuit, we have

$$-L \frac{dI}{dt} - IR = 0 .$$

The first term is the EMF due to the inductor,  $-LdI/dt$ . Why the minus sign? By Lenz’s law, the inductor always fights change. If the current “wants” to be reduced, the sign of the inductor’s EMF needs to fight that. The second term is the usual voltage drop across a resistor. Rearranging leads to the easily integrated equation

$$\frac{dI}{I} = -\frac{R dt}{L} ,$$

which in turn leads to the exponential solution

$$I(t) = I(t = 0)e^{-Rt/L} .$$

(Note that we have redefined the meaning of  $t = 0$  — this is now the time at which we open  $S1$  and close  $S2$ .)

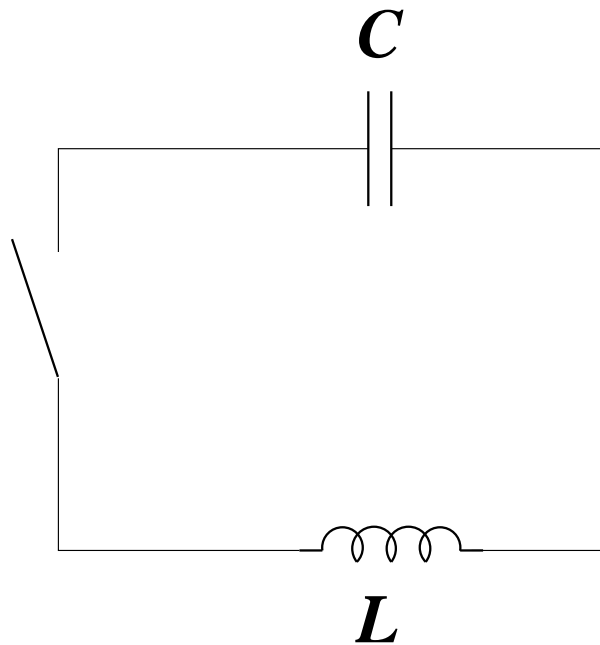
When thinking about *any* circuit with inductance, the key thing you should keep in mind is that inductors cause currents to have a kind of “inertia”. If there is no current flowing, the inductor will force the current to build up gradually; if there is current flowing, the inductor will force it to change gradually. If current is already flowing and there is a sudden break in the circuit, the inductor will do whatever it takes to keep the current flowing. A consequence of this is that when we open switches or break circuits very suddenly, we may see a large spark! That spark is because of Lenz’s law — inductors in the circuit refuse to let the flux through them change. If that means that they must force the current to flow through the air, creating a huge, fiery spark — so be it.

### 16.1.3 Unit check

Before moving on, let’s do the traditional sanity check on our units: the ratio  $L/R$  should have the dimension of time for any of the above results to make sense. In cgs, the units of  $L$  are  $\text{sec}^2/\text{cm}$ ; the units of  $R$  are  $\text{sec}/\text{cm}$ . So  $L/R$  is indeed a time measured in seconds. In SI, the unit of  $L$  is the Henry, which is Volt-sec/Amp; the unit of  $R$  is the Ohm, which is Volt/Amp. So  $L/R$  is again a time measured in seconds. Three cheers for dimensional analysis.

## 16.2 LC circuits

Exponential decay (or saturation) is dull. We’re now going to look at a circuit configuration that does something new. Consider what happens when we connect an inductor to a capacitor:



Suppose that the capacitor is charged up to a charge separation  $Q_0$ . We close the switch at  $t = 0$ . How does the circuit behave after this?

As usual, our governing analysis of the system comes from Kirchhoff/Faraday: going around the circuit, we have

$$\frac{Q}{C} - L \frac{dI}{dt} = 0$$

where  $Q$  is the charge on the capacitor, and  $I$  is the current flowing through the inductor. To get anywhere with this, we of course need to relate the current flowing to the charge on the capacitor. Since the current comes from charge flowing off the capacitor, we have

$$I = -\frac{dQ}{dt}.$$

We thus finally end up with the differential equation

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0.$$

Differential equations of this form are best solved via an age old technique: guessing. In particular, guided by our experience with other systems that have a similar form, we guess that the solution has the form

$$Q(t) = A \cos \omega_0 t + B \sin \omega_0 t.$$

We plug this solution back into our differential equation; the key thing we need is

$$\begin{aligned} \frac{d^2Q}{dt^2} &= -\omega_0^2 A \cos \omega_0 t - \omega_0^2 B \sin \omega_0 t \\ &= -\omega_0^2 Q. \end{aligned}$$

Our assumed solution works fine provided we set the angular frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} .$$

(You should be able to verify immediately that this has the correct units.) The constants  $A$  and  $B$  are free to be adjusted to match our initial conditions — the value of the charge on the capacitor  $Q$  at  $t = 0$ , and the current flowing in the circuit at  $t = 0$ :

$$\begin{aligned} Q(0) &= Q_0 \\ &= A \cos(0) + B \sin(0) \\ \rightarrow A &= Q_0 . \end{aligned}$$

$$\begin{aligned} I(0) &= - \left( \frac{dQ}{dt} \right)_{t=0} = 0 \\ &= -\omega_0 A \sin(0) + \omega_0 B \cos(0) \\ \rightarrow B &= 0 . \end{aligned}$$

We now have the full solution to this problem: the charge on the capacitor, as a function of time, is

$$Q(t) = Q_0 \cos \omega_0 t .$$

Equivalently, the current flowing in the circuit,  $I(t) = -dQ/dt$ , is

$$\begin{aligned} I(t) &= \omega_0 Q_0 \sin \omega_0 t \\ &= \frac{Q_0}{\sqrt{LC}} \sin \omega_0 t . \end{aligned}$$

This sinusoidal behavior is the generic feature of LC circuits. Notice that the charge has maximum magnitude when the current is zero, and vice versa — the current and the charge are “90° out of phase” with each other.

A nice way to make sense of the current and charge behavior is by looking at the energetics of the circuit elements. The energy stored in the capacitor, as a function of time, is

$$\begin{aligned} U_C(t) &= \frac{Q(t)^2}{2C} \\ &= \frac{Q_0^2}{2C} \cos^2 \omega_0 t . \end{aligned}$$

The energy stored in the inductor, as a function of time, is

$$\begin{aligned} U_L(t) &= \frac{1}{2} L I(t)^2 \\ &= \frac{1}{2} \frac{1}{LC} L Q_0^2 \sin^2 \omega_0 t \\ &= \frac{Q_0^2}{2C} \sin^2 \omega_0 t . \end{aligned}$$

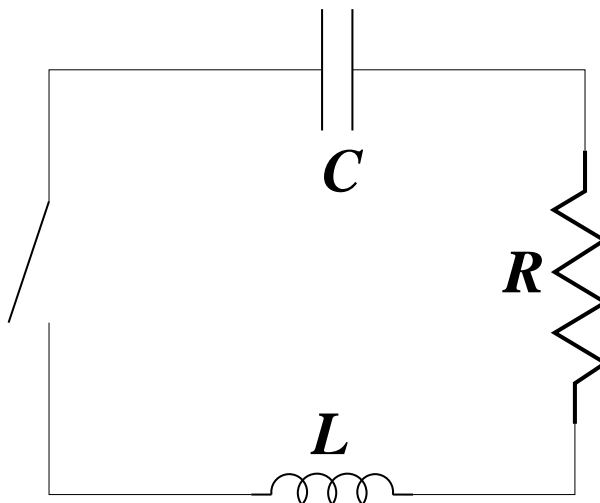
The energy in the inductor and the energy in the capacitor each oscillate. Note that their sum is constant, and is exactly equal to the total energy that was initially stored in the capacitor:

$$\begin{aligned} U_{\text{tot}} &= U_C(t) + U_L(t) = \frac{Q_0^2}{2C} (\cos^2 \omega_0 t + \sin^2 \omega_0 t) \\ &= \frac{Q_0^2}{2C}. \end{aligned}$$

Conservation of energy works! In this case, there's an even cooler interpretation. The energy in the capacitor is stored as electric fields; that in the inductor is stored as magnetic fields. What we are seeing here is that the system's total energy is just sloshing back and forth, from electric to magnetic to electric to ... This circuit is just an electromagnetic swing set!

### 16.3 RLC circuits

The circuit we just discussed is an interesting one ... but it's also somewhat fictional, since we are idealizing it as having no resistance. Since *all* circuits have *some* resistance<sup>1</sup>, the RLC circuit sketched here is *far* more relevant to real world applications than the LC circuit:



How would we expect this circuit to behave? Our intuition on the energetics of LC circuits is helpful here: qualitatively, we should expect the same “sloshing” of energy between the inductor and capacitor. However, we now expect that some energy will be dissipated through each cycle because of the circuit's resistance. Our solution for the charge and the current in the circuit should look like a decaying oscillation of some kind.

To make progress, we need to do some calculating. We again assume that the capacitor initially has some charge  $Q_0$  on it, and that we close the switch at  $t = 0$ . Faraday/Kirchhoff then tells us how to set up the equations for this circuit:

$$\frac{Q}{C} - IR - L \frac{dI}{dt} = 0$$

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<sup>1</sup>Unless they are superconducting, but that opens a whole new barrel of worms ...

which reduces to

$$\frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{LC} Q = 0 .$$

The usual way to solve these equations is the tried and true friend of the physicist, inspired guessing: we assume that the solution takes the form

$$Q(t) = e^{-t/\tau} (A \cos \omega t + B \sin \omega t)$$

— very similar to the form we found for the LC circuit. The major new feature is the exponentially decaying prefactor. We could then plug this assumed form of  $Q(t)$  into the differential equation and then grind out a lot<sup>2</sup> of algebra. After the smoke clears, the algebra will tell you that, for this solution to work,  $\tau$  and  $\omega$  must take on certain values. The constants  $A$  and  $B$  will be left as adjustable parameters to be used for matching initial conditions.

Feel free to pursue this route on your own! We will instead use this moment as an opportunity to introduce a different way of thinking about this problem: we will use complex exponentials. For the remainder of this lecture, I will assume that you are familiar with the supplementary notes on complex numbers I put together (available under “Handouts” on the 8.022 webpage).

The key idea is to take the solution  $Q(t)$  to be the real part of a complex function  $\tilde{Q}(t)$ :

$$Q(t) = \text{Re} [\tilde{Q}(t)] .$$

Then, as long as the resistance, capacitance, and inductance are themselves real numbers, the *complex* charge  $\tilde{Q}$  is also a solution to the differential equation:

$$\frac{d^2\tilde{Q}}{dt^2} + \frac{R}{L} \frac{d\tilde{Q}}{dt} + \frac{1}{LC} \tilde{Q} = 0 .$$

Our assumption is that

$$\tilde{Q} = A e^{i\phi_0} e^{i\alpha t} .$$

As we will see shortly,  $A$  and  $\phi_0$  (the “amplitude” and the “phase”) are the constants that we adjust to match initial conditions. We will now work out what  $\alpha$  is. Bear in mind that  $\alpha$  may *itself* be a complex number!!!

To proceed, we need to know the derivatives of  $\tilde{Q}$ . This is where the magic of using complex exponentials for solving differential equations comes in: This technique turns *differential* equations into *algebraic* ones. In particular, we have

$$\begin{aligned} \frac{d\tilde{Q}}{dt} &= i\alpha \tilde{Q} \\ \frac{d^2\tilde{Q}}{dt^2} &= -\alpha^2 \tilde{Q} . \end{aligned}$$

Plugging this back into the governing differential equation, we get

$$\tilde{Q} \left( -\alpha^2 + i\alpha \frac{R}{L} + \frac{1}{LC} \right) = 0$$

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<sup>2</sup>A LOT.

which means that  $\alpha$  must satisfy

$$\alpha^2 - i\alpha \frac{R}{L} - \frac{1}{LC} = 0 .$$

This is just a quadratic equation, whose solution is

$$\alpha = i \frac{R}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} .$$

We thus have two solutions for  $\tilde{Q}$ :

$$\begin{aligned} \tilde{Q}_+ &= Ae^{i\phi_0} e^{-Rt/(2L)} e^{+it\sqrt{1/(LC) - R^2/(4L^2)}} \\ \tilde{Q}_- &= Ae^{i\phi_0} e^{-Rt/(2L)} e^{-it\sqrt{1/(LC) - R^2/(4L^2)}} . \end{aligned}$$

The distinction between these two solutions is physically meaningless since they have the same real part! We might as well just keep one (the + solution, say) and be done with it. Taking the real part, our final solution becomes

$$Q(t) = Ae^{-Rt/(2L)} \cos(\omega t + \phi_0)$$

where

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} .$$

Although the use of complex numbers may seem kind of counterintuitive here, it makes solving this problem almost trivial. Rather than needing to fight through trig functions and their derivatives, you just need to solve one quadratic equation.

Complex numbers make for simple analysis.

### 16.3.1 The “weak damping” limit

An important limit of the above circuit is that of *weak damping*. This holds when the resistance is “small”, in a well-defined way. To see what constitutes weak damping, we compute the current from our charge solution:

$$\begin{aligned} I(t) &= -\frac{dQ}{dt} \\ &= Ae^{-Rt/(2L)} \left[ \omega \sin(\omega t + \phi_0) + \frac{R}{2L} \cos(\omega t + \phi_0) \right] . \end{aligned}$$

The current consists of the sum of two different sinusoidal functions. The amplitude of this first term is proportional to the frequency  $\omega$ ; the amplitude of the second is proportional to the damping rate  $R/(2L)$ .

Weak damping means that the frequency is much larger than the damping rate:

$$\omega \gg \frac{R}{2L} .$$

If this is the case, then the second term in the current may be ignored, to high accuracy, relative to the first term:

$$I(t) \simeq Ae^{-Rt/(2L)} \omega \sin(\omega t + \phi_0) .$$



We can actually make one further useful simplification in this limit: if  $\omega \gg R/2L$ , then you should be able to convince yourself quite easily that

$$\begin{aligned}\omega &\equiv \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \\ &\simeq \sqrt{\frac{1}{LC}} \\ &= \omega_0 .\end{aligned}$$

In other words, when we have weak damping, the oscillation frequency is — to high accuracy — the same frequency as we would have found if the resistor were not present!

In the weak damping limit, it is now simple to match the constants  $A$  and  $\phi_0$  to our initial conditions:

$$\begin{aligned}Q(t=0) &= Q_0 \\ &= A \cos(\phi_0) \\ I(t=0) &= 0 \\ &= A\omega_0 \sin(\phi_0) .\end{aligned}$$

The solution to these equations is

$$\begin{aligned}A &= Q_0 \\ \phi_0 &= 0 .\end{aligned}$$

Thus, our “full” solution, in the weak damping limit, for this circuit is

$$\begin{aligned}Q(t) &= Q_0 e^{-Rt/(2L)} \cos \omega_0 t \\ I(t) &= \omega_0 Q_0 e^{-Rt/(2L)} \sin \omega_0 t \\ \omega_0 &= \sqrt{\frac{1}{LC}} .\end{aligned}$$

A similar calculation can be done without assuming weak damping, but the details are much messier.

### 16.3.2 The quality factor

In the weak damping limit, the energy content of the circuit behaves in a very nice way. The energy of the capacitor is given by

$$\begin{aligned}U_C(t) &= \frac{Q(t)^2}{2C} \\ &= \frac{Q_0^2}{2C} e^{-Rt/L} \cos^2 \omega_0 t .\end{aligned}$$

The energy of the inductor is given by

$$\begin{aligned}U_L(t) &= \frac{1}{2} L I(t)^2 \\ &= \frac{1}{2} \omega_0^2 L Q_0^2 e^{-Rt/L} \sin^2 \omega_0 t \\ &= \frac{Q_0^2}{2C} e^{-Rt/L} \sin^2 \omega_0 t .\end{aligned}$$

(In going from the second to the third line, I've used  $\omega_0^2 = 1/LC$ .) The sum of these energies is

$$\begin{aligned} U_{\text{tot}}(t) &= U_C(t) + U_L(t) \\ &= \frac{Q_0^2}{2C} e^{-Rt/L} (\cos^2 \omega_0 t + \sin^2 \omega_0 t) \\ &= \frac{Q_0^2}{2C} e^{-Rt/L} . \end{aligned}$$

Since  $Q_0^2/2C$  is the initial energy stored in the system, this means that the *total* energy content just exponentially damps away. (If we are not in the weak damping limit, a similar result holds, though the details are not quite as nice.)

Scientists and engineers like to be able to quantify how “good” an RLC circuit is, and so we invented a quantity called the circuit’s quality  $Q$  (not to be confused with its charge). A circuit with high quality is one that oscillates many times before it loses a significant amount of energy. To make this somewhat vague notion precise, the following definition is typically adopted: “The quality factor  $Q$  is the number of radians through which a damped system oscillates by the time that its energy content has fallen by a factor of  $1/e$ .”

This may sound a little odd, but it deconstructs into a very simple mathematical statement. First, how much time does it take for the energy to fall by a factor of  $1/e$ ? By the above discussion, this is easy:

$$T(\text{energy to fall by } 1/e) = L/R .$$

Plug this time into the energy stuff discussed above, and you see that the energy is at  $1/e$  of its initial level.

How many radians does the system oscillate through in this time? We just multiply the time  $T$  by the system’s angular frequency  $\omega \simeq \omega_0$ ! The quality factor of an RLC circuit is

$$\begin{aligned} Q &= \frac{\omega L}{R} \\ &\simeq \frac{\omega_0 L}{R} . \end{aligned}$$

As we will see when we begin exploring AC circuits, there is another way to understand the quality factor of circuits, related to how a circuit behaves “on resonance” (and very close to resonance). We will define the quality factor in what seems to be a completely different way, and will end up with exactly the same result! This is not an accident of course, but is deeply connected to how energy is stored and dissipated in circuits.

The concepts we hit in this lecture — exponential decay of an oscillation; quality factor — and that we’ll hit in the next few lectures (e.g., resonance) pop up in an enormous number of contexts. For example, the quality factor of a distorted black hole — the number of radians worth of gravitational waves produced by the oscillations of its distorted event horizon emitted as it settles down to an equilibrium state — is given approximately by

$$Q_{\text{BH}} \simeq 2 \left( 1 - \frac{c}{G} \frac{S}{M^2} \right)^{-9/20}$$

(where  $G$  is the gravitational constant,  $c$  is the speed of light,  $S$  is the magnitude of the black hole’s spin angular momentum, and  $M$  is its mass). There are many other examples of quality factors that pop up repeatedly in science and engineering; you *will* see many of these notions again!