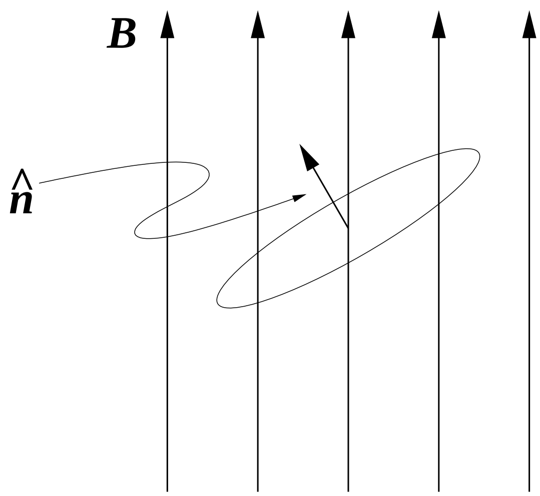
Massachusetts Institute of Technology
Department of Physics
8.022 Spring 2005

Lecture 17:
AC Circuits; Impedance.

17.1 AC circuits: Intro

As mentioned in Lecture 15, we *use* induction to generate the vast majority of electrical power in the world. Typically, we do this with some variation on the "vary the dot product" mechanism:



The loops of wire are rotated by some external source of work — a diesel engine, a waterfall, nuclear produced steam, etc. The magnetic field largely acts as a mechanism to transform that external work into electrical power. Doing so, we end up with a driving EMF that is sinusoidal:

$$\Phi_B = BA\cos\omega t$$
$$\longrightarrow \mathcal{E} = \frac{\omega BA}{c}\sin\omega t\ .$$

At least in part for this reason, the electricity that we get "from the wall" oscillates sinusoidally. To insure some degree of uniformity, standard choices for the oscillation frequency have been agreed upon. To demonstrate that humans are illogical beings who will shoot themselves in the foot at any opportunity, two such standards are commonly used:

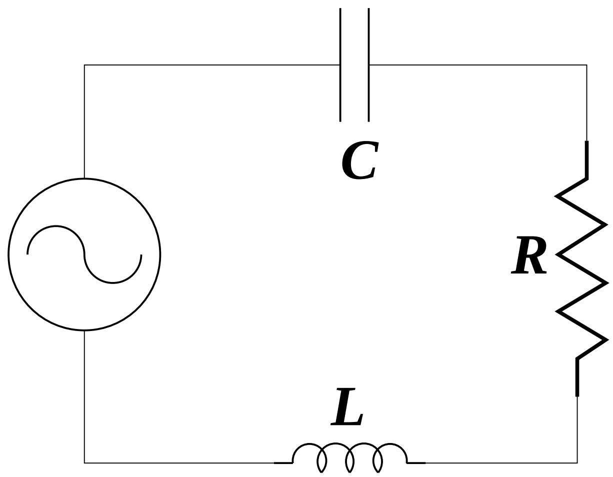
$$\omega = 2\pi \times 60\,\mathrm{Hz}$$
$$\omega = 2\pi \times 50\,\mathrm{Hz}\ .$$

60 Hz is used in North America, much of South America, South Korea, and western Japan; 50 Hz is used in Europe, most of Africa (except, for example, Liberia), the rest of South America,

and most of Asia (including eastern Japan). See http://kropla.com/electric2.htm for a summary.

The main reason for going through this is to motivate how *essential* it is to understand how alternating current (AC) circuits work. Having spent as much time as we have studying *direct current* (DC) circuits, generalizing to AC is not too difficult — particularly if we are smart enough to recognize that our life is made a LOT easier by working with complex valued quantities.

Our goal in this lecture will be to understand how to find the current which flows in the following circuit:



The symbol on the left is a source of AC EMF. This device supplies an EMF to the rest of the circuit which we will take to be

$$\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t \ .$$

In much of our analysis, we will find it *very* useful to take this be the real part of a complex EMF $\tilde{\mathcal{E}}(t)$:

$$\mathcal{E}(t) = \mathrm{Re}\left[\tilde{\mathcal{E}}(t)\right]$$

where

$$\tilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t} \ .$$

(Note that $\mathcal{E}_0$ is a purely real number.)

We will analyze this circuit by just generalizing the operating principles that we used for DC circuits. In particular,

1. The sum of the voltage drops in the loop must equal the driving EMF. This is true for both the real form of the voltage

$$\mathcal{E}(t) = V_R(t) + V_C(t) + V_L(t)$$

and the complex form:

$$\tilde{\mathcal{E}}(t) = \tilde{V}_R(t) + \tilde{V}_C(t) + \tilde{V}_L(t) \ .$$

You may ask "What is the voltage drop across an inductor — doesn't an inductor act as a *source* of EMF?" Indeed, you would be correct for asking this. To keep things uniform, what we do is move the EMF from the left-hand side to the right-hand side, introducing a minus sign. Hence, $V_L(t) = -\mathcal{E}_{\rm ind} = L dI/dt$. There's nothing deep here — it's just a trick we introduce in order to treat all circuit elements in a uniform way.

2. The same current must pass through every circuit element. Likewise true both for real and for complex:

$$I(t) = I_R(t) = I_C(t) = I_L(t)$$
$$\tilde{I}(t) = \tilde{I}_R(t) = \tilde{I}_C(t) = \tilde{I}_L(t) \; .$$

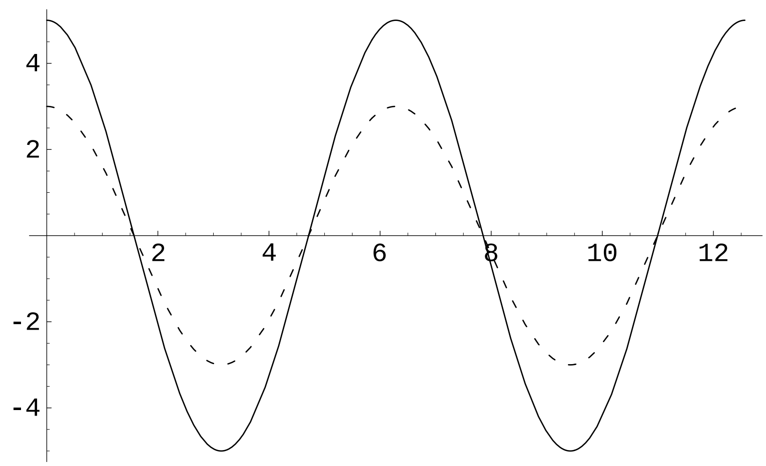
To understand this "driven RLC" circuit, we will first analyze the response of each of its circuit elements one by one.

17.1.1 AC + resistor

Resistors obey Ohm's law. Period. End of story. If we hook our driving AC EMF source across a resistor R, we have

$$\mathcal{E}(t) = V_R(t) = I(t) R \; .$$

At the risk of totally beating this to death, if we plot our driving EMF and the current in the circuit together, the plot generally looks like this:

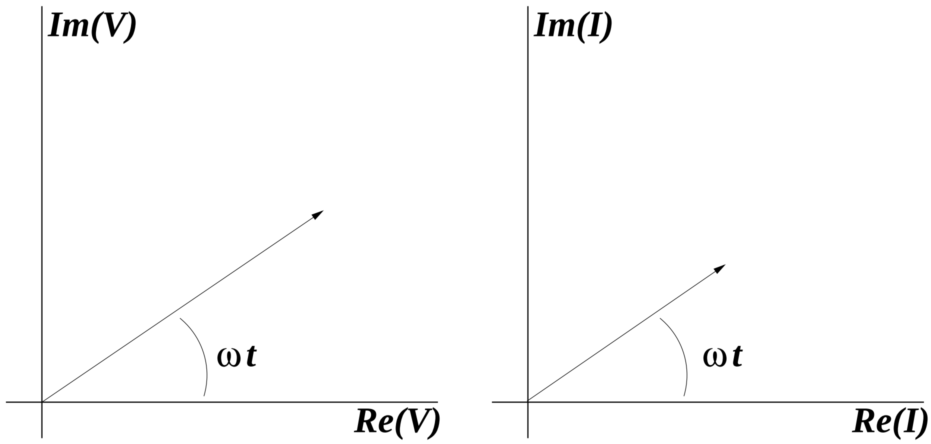


(The solid line is the driving EMF, the dotted line is the current.) The key thing to take away is the following rule:

In a resistor, the current and the voltage are IN PHASE with each other.

In other words, the peak voltage occurs at the same time as the peak current, the minimum voltage coincides with the minimum current, etc.

The same information represented by phasors in the complex plane is



"In phase" means that both phasors are at the same angle (though their magnitudes, or lengths, may be different).

17.1.2 AC + capacitor

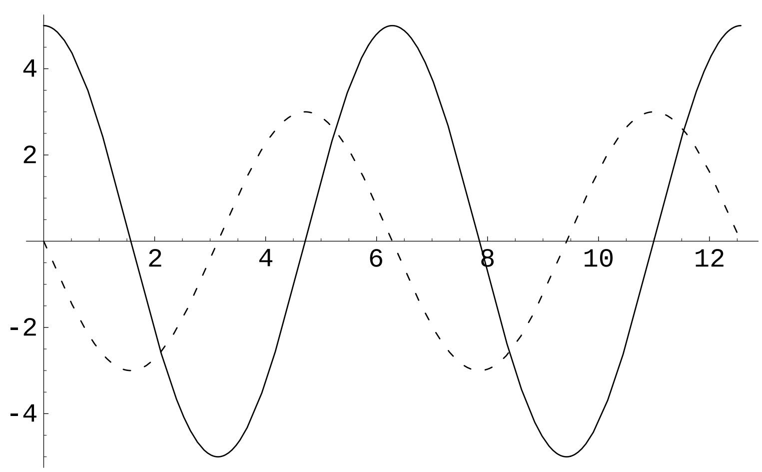
The situation is not quite so straightforward if we hook our AC EMF source across a capacitor. The governing equation in this case is

$$\mathcal{E}(t) = V_C(t) = \frac{Q(t)}{C} \; .$$

Plugging in $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$, solving for $Q(t)$ and taking the derivative, we have

$$\begin{aligned} I(t) &= -\omega C \mathcal{E}_0 \sin \omega t \\ &= \omega C \mathcal{E}_0 \cos(\omega t + \pi/2) \; . \end{aligned}$$

Let's plot the voltage and the current together:



Notice that maxima in the current occur before maxima in the voltage (likewise for minima):

Current LEADS voltage by 90°. Equivalently, voltage LAGS current by 90°.

The relation between the current and the voltage in the capacitor has a particularly nice form when expressed using complex numbers. The voltage is given by

$$V_C(t) = \mathcal{E}_0 \cos(\omega t) = \mathrm{Re}\left[\tilde{V}_C(t)\right]$$

where

$$\tilde{V}_C(t) = \mathcal{E}_0 e^{i\omega t} \ .$$

The current is given by

$$I(t) = \omega C \mathcal{E}_0 \cos(\omega t + \pi/2) = \mathrm{Re}\left[\tilde{I}(t)\right]$$

where

$$\begin{aligned} \tilde{I}(t) &= \omega C \mathcal{E}_0 e^{i(\omega t + \pi/2)} \\ &= i\omega C \mathcal{E}_0 e^{i\omega t} \ . \end{aligned}$$

(On the last line I have used the fact that $e^{i\pi/2} = i$.)

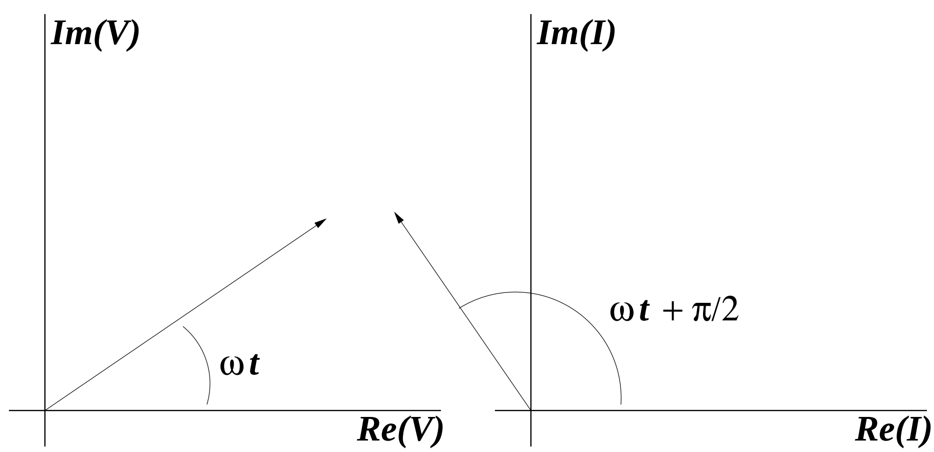
Putting all of this together, we can write the relation between the voltage and the current in the capacitor in a form that is just Ohm's law:

$$\tilde{V}_C(t) = \tilde{I}(t) Z_C$$

where

$$Z_C = \frac{1}{i\omega C}$$

is the *impedance* of the capacitor. Viewed as phasors in the complex plane, the current and the voltage look as follows:



Notice that the current is rotated 90° ahead of the voltage, in keeping with the "current leads voltage, voltage lags current" rule.

17.1.3 AC + inductor

Now, let's hook our AC EMF across an inductor. Our governing equation is clearly going to be

$$\mathcal{E}(t) = V_L(t) = L\frac{dI}{dt} \ .$$

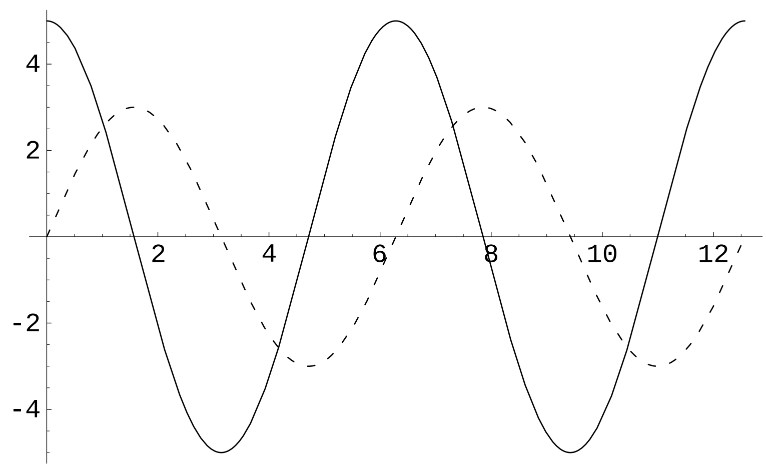
Plugging in $\mathcal{E}(t) = \mathcal{E}_0 \cos\omega t$, we find

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L}\cos\omega t \ .$$

Integrating up, we find

$$\begin{aligned} I(t) &= \frac{\mathcal{E}_0}{\omega L}\sin\omega t \\ &= \frac{\mathcal{E}_0}{\omega L}\cos\left(\omega t - \pi/2\right) \ . \end{aligned}$$

Let's plot the voltage and the current together:



In this case, maxima in the current occur *after* maxima in the voltage (likewise for minima):

Current LAGS voltage by 90°. Equivalently, voltage LEADS current by 90°.

The relation between the inductor's current and voltage again has a particularly nice form when expressed with complex numbers. The voltage is

$$V_L(t) = \mathcal{E}_0\cos(\omega t) = \mathrm{Re}\left[\tilde{V}_L(t)\right]$$

where

$$\tilde{V}_L(t) = \mathcal{E}_0 e^{i\omega t} \ .$$

The current is given by

$$I(t) = \frac{\mathcal{E}_0}{\omega L}\cos(\omega t - \pi/2) = \mathrm{Re}\left[\tilde{I}(t)\right]$$

where

$$\tilde{I}(t) = \frac{\mathcal{E}_0}{\omega L} e^{i(\omega t - \pi/2)}$$
$$= \frac{\mathcal{E}_0}{i\omega L} e^{i\omega t} .$$

(On the last line I have used the fact that $e^{-i\pi/2} = 1/i$.)

Putting all of this together, we can write the relation between the voltage and the current in the inductor in a form that is just Ohm's law:

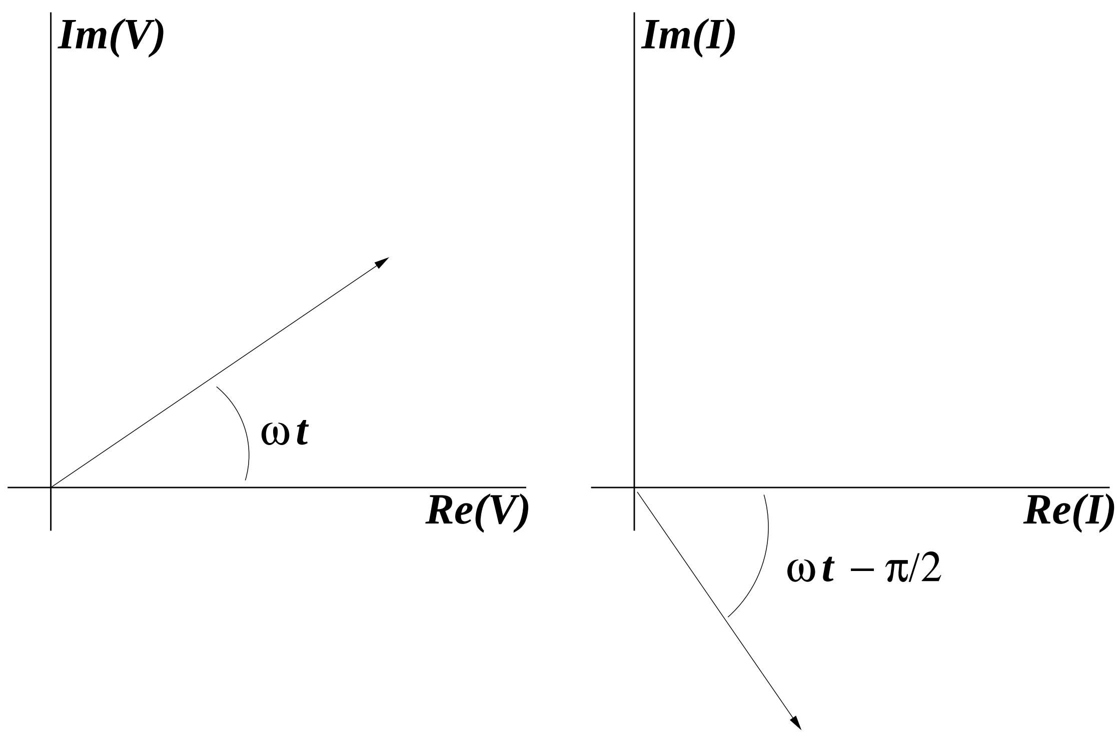
$$\tilde{V}_L(t) = \tilde{I}(t) Z_L$$

where

$$Z_L = i\omega L$$

is the impedance of the inductor.

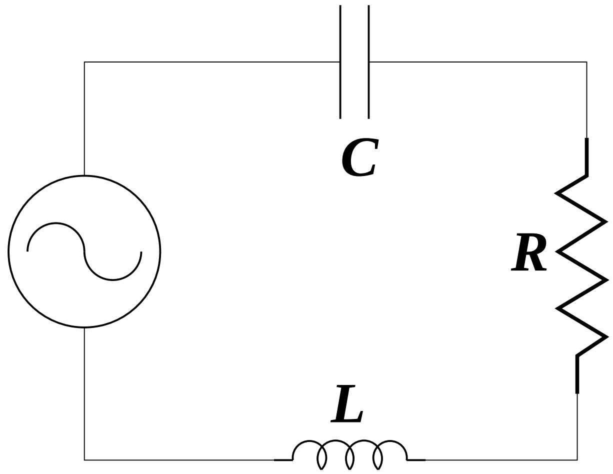
Viewed as phasors in the complex plane, the current and the voltage are



Notice that the current is 90° behind in the voltage, in keeping with the notion that current lags voltage in inductors.

17.2 Putting it together: Driven series RLC, take 1

We're now finally ready to analyze this circuit:



Doing so is actually quite easy as long as we work in the complex domain! We just use the rules we established earlier,

$$\tilde{\mathcal{E}}(t) = \tilde{V}_R(t) + \tilde{V}_L(t) + \tilde{V}_C(t)$$
$$\tilde{I}(t) = \tilde{I}_R(t) = \tilde{I}_L(t) = \tilde{I}_C(t)$$

to which we now add our new version of Ohm's law:

$$\tilde{V}_X(t) = \tilde{I}_X(t) Z_X$$

where X stands for the resistor, the capacitor, or the inductor.

Our first equation becomes

$$\begin{aligned} \mathcal{E}_0 e^{i\omega t} &= \tilde{I}(t)\left[R + i\omega L + \frac{1}{i\omega C}\right] \\ &= \tilde{I}(t)\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] \\ &= \tilde{I}(t) Z_{\text{tot}} \end{aligned}$$

where Z_{tot} is the *total* impedance of the circuit:

$$Z_{\text{tot}} = R + i\omega L - \frac{i}{\omega C} = R + i\left(\omega L - \frac{1}{\omega C}\right) .$$

Notice that the total impedance is dominated by the inductor at high frequency, and by the capacitor at low frequency. We will see this again very soon.

Using the above result, the *complex* current in the circuit is given by

$$\tilde{I}(t) = \frac{\mathcal{E}_0 e^{i\omega t}}{R + i\left[\omega L - 1/(\omega C)\right]} .$$

To keep our notation simple, we will write this as

$$\tilde{I}(t) = I_0 e^{i\omega t} e^{-i\phi} .$$

In other words, we are just factoring out the $e^{i\omega t}$ piece, and putting

$$I_0 e^{-i\phi} = \frac{\mathcal{E}_0}{R + i\left[\omega L - 1/(\omega C)\right]} \; .$$

Don't get excited about the minus sign in front of the $i\phi$. All this means is that we *define* ϕ as the angle by which the current will **LAG** the voltage. This is convention, not anything deep. If ϕ turns out to be negative, then it means that the current actually leads the voltage.

Our job now will be to figure out the "current amplitude" I_0 and the current phase angle ϕ. To make progress on this, we note that *any* complex number can be written in the form

$$z = re^{-i\phi}$$

where $r = \sqrt{zz^*}$. Let's look at the magnitude of the current first:

$$\begin{aligned}
|\tilde{I}| = I_0 &= \sqrt{\tilde{I}\tilde{I}^*} \\
&= \mathcal{E}_0 \sqrt{\left(\frac{1}{R + i\left[\omega L - 1/(\omega C)\right]}\right)\left(\frac{1}{R - i\left[\omega L - 1/(\omega C)\right]}\right)} \\
&= \frac{\mathcal{E}_0}{\sqrt{R^2 + \left[\omega L - 1/(\omega C)\right]^2}} \; .
\end{aligned}$$

This is pretty cool. The amount of current that flows through this circuit varies — quite strongly! — as a function of the AC frequency ω. Notice that the *maximum* current amplitude occurs when $\omega = \omega_0 = 1/\sqrt{LC}$: at this frequency, the L and the C terms cancel each other out, leaving just the resistance term.

Now we need to figure out that phase term. To set things up, note that from the rule

$$z = re^{-i\phi} = r\left(\cos\phi - i\sin\phi\right)$$

we can extract

$$\tan\phi = -\frac{\mathrm{Im}\left[z\right]}{\mathrm{Re}\left[z\right]} \; .$$

Now, we want to do something like this for our complex current. Let us massage this relation a bit:

$$\begin{aligned}
I_0 e^{-i\phi} &= \frac{\mathcal{E}_0}{R + i\left[\omega L - 1/(\omega C)\right]} \\
&= \frac{\mathcal{E}_0}{R + i\left[\omega L - 1/(\omega C)\right]}\left(\frac{R - i\left[\omega L - 1/(\omega C)\right]}{R - i\left[\omega L - 1/(\omega C)\right]}\right) \\
&= \frac{\mathcal{E}_0}{R^2 + \left[\omega L - 1/(\omega C)\right]^2}\left(R - i\left[\omega L - 1/(\omega C)\right]\right) \; .
\end{aligned}$$

Taking the ratio of the imaginary part to the real part, we have

$$\begin{aligned}\tan\phi &= \frac{\omega L - 1/(\omega C)}{R} \\ &= \frac{\omega L}{R} - \frac{1}{\omega C R}\,.\end{aligned}$$

Typically, ϕ ends up having a shape as a function of ω that looks something like this:

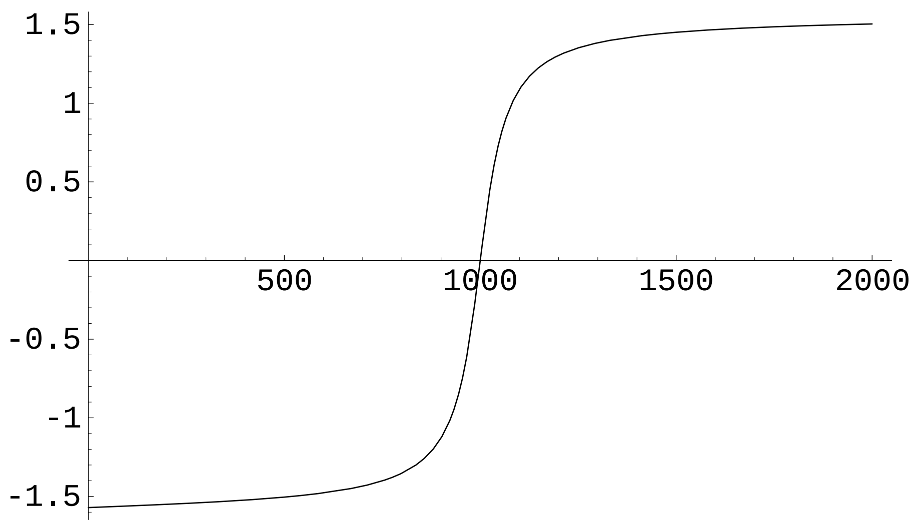
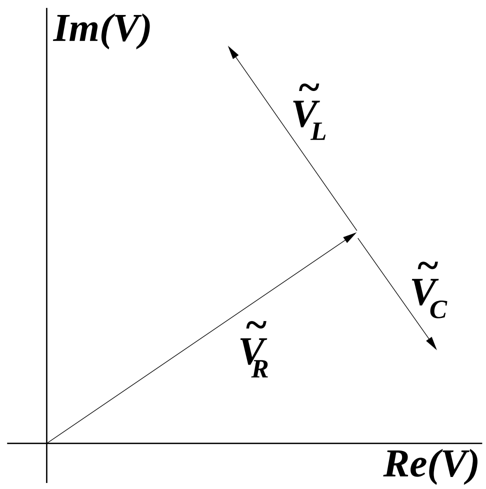


Figure 1: ϕ vs ω for $R = 1$ Ohm, $L = 0.01$ Henry, $C = 10^{-4}$ Farad.

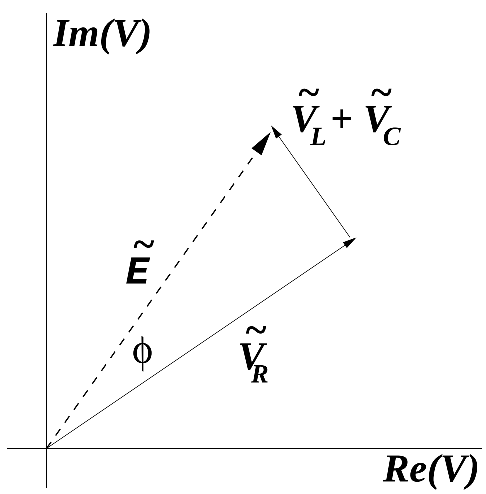
Notice that, for low frequency, the phase angle limits to $-\pi/2$: the current *LEADS* the voltage by $\sim 90°$ at low frequency. This is exactly the behavior we saw when we considered our EMF attached to just the capacitor! This has a very nice interpretation: at low frequencies, the capacitor dominates the overall impedance of the circuit — exactly what we expected on intuitive grounds from the form of $Z_{\rm tot}$. Likewise, at high frequencies, the phase angle limits to $+\pi/2$: the current *LAGS* voltage by $\sim 90°$ — exactly the behavior we see for EMF attached to the inductor; and, exactly what $Z_{\rm tot}$ would suggest.

17.3 Putting it together: Driven series RLC, take 2

The phasor picture allows us to put together another way of getting the above result that is perhaps a little more intuitive. The basic idea is to treat the phasors as vectors in the complex plane. Then, we just add them all head to tail:



This diagram shows the voltage drop across the resistor, $\tilde{V}_R$, along with the drop across the inductor (90° ahead) and across the capacitor (90° behind). This drawing simplifies by combining the voltage drops across the inductor and the capacitor:



We can now work out a lot of things using simple geometry and trigonometry! For example, the rule that the total voltage drop in the circuit must equal the driving EMF tells us that

$$\tilde{V}_R + \tilde{V}_L + \tilde{V}_C = \tilde{\mathcal{E}} \ .$$

The driving EMF $\tilde{\mathcal{E}}$ is the "resultant" vector we obtain by summing the other three phasors; ϕ is the angle between $\tilde{\mathcal{E}}$ and $\tilde{V}_R$ (since $\tilde{V}_R$ is in phase with the current in the circuit).

With the Pythagorean theorem, we get the current's magnitude very easily:

$$\mathcal{E}_0^2 = V_R^2 + (V_L - V_C)^2$$

$$
\begin{aligned}
&= I_0^2\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right] \\
\longrightarrow I_0 &= \frac{\mathcal{E}_0}{\sqrt{R^2 + [\omega L - 1/(\omega C)]^2}} \, .
\end{aligned}
$$

Likewise, we can get the phase angle ϕ with simple trigonometry:

$$
\begin{aligned}
\tan\phi &= \frac{V_L - V_C}{V_R} \\
&= \frac{\omega L}{R} - \frac{1}{\omega C R} \, .
\end{aligned}
$$

Exactly what we had before!

17.4 Summary

AC circuits are surprisingly easy to analyze, provided you follow some simple rules:

1. Do all of your calculations using complex voltages and currents. Replace $\cos(\omega t)$ with $e^{i\omega t}$; replace $\sin(\omega t)$ with $\cos(\omega t - \pi/2) = e^{i(\omega t - \pi/2)}$.

2. Relate the voltage drops and currents across any circuit element using our "generalized Ohm's law":

   $$\tilde{V}_X = \tilde{I}_X Z_X \, .$$

   The quantity Z_X is the impedance of a circuit element X. Impedence is just resistance on steroids: it works exactly like resistance in a DC circuit, but is complex, frequency dependent, and works in AC circuits. The impedance formulas you need to know are

   $$
   \begin{aligned}
   Z_R &= R \\
   Z_C &= 1/(i\omega C) \\
   Z_L &= i\omega L \, .
   \end{aligned}
   $$

3. Analyze your circuit using impedance and complex voltages and currents just like you would analyze a DC circuit containing only resistors. Be careful to get all magnitudes and phases correct — the math is likely to be a little more hairy than a simple DC circuit. A good phasor diagram can help organize things.

4. Take the real parts of all relevant quantities at the end of the day.

And that's it.