Scott Hughes *21 April 2005*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LECTURE 18:
AC CIRCUITS.
POWER AND ENERGY; RESONANCE. FILTERS.

18.1 AC circuits: Recap and summary

Last time, we looked at AC circuits and found that they are quite simple to analyze provided we follows some simple rules:

1. Work with complex valued voltages and currents. Our driving AC EMF is usually something like $\mathcal{E}(t) = \mathcal{E}_0 \cos\omega t$; replace this with $\tilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t}$ (so that $\mathcal{E}(t) = \mathrm{Re}[\tilde{\mathcal{E}}_0(t)]$).

2. The voltage drop across any circuit element obeys a generalized, complex version of Ohm's law:

$$\tilde{V}_X = \tilde{I}_X Z_X$$

where Z_X is the impedance of circuit element X. Impedance works just like resistance, but is complex and frequency dependent:

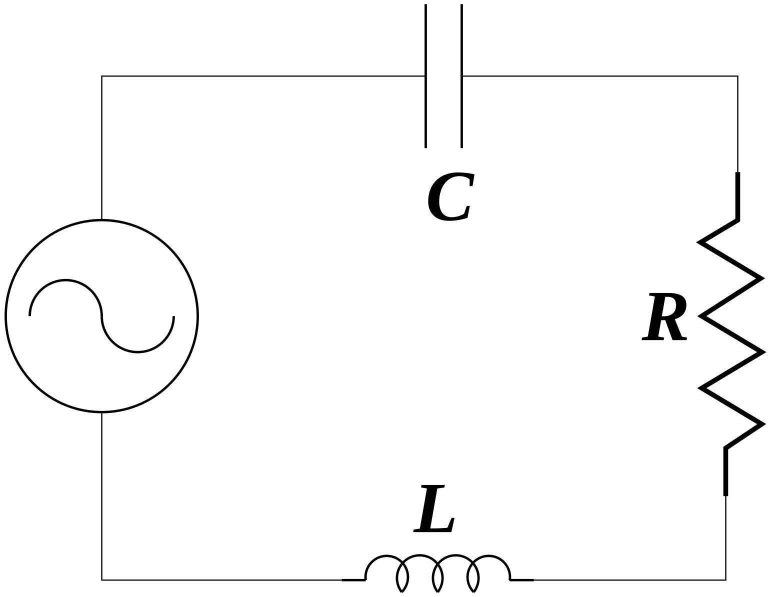
$$\begin{aligned} Z_R &= R \\ Z_C &= 1/(i\omega C) \\ Z_L &= i\omega L \; . \end{aligned}$$

3. Analyze the circuit as though it were a simple DC circuit containing only resistors as circuit elements. You may find phasor diagrams helpful for making sure that you get the magnitudes and phases correct.

4. Take the real part at the end of the day.

18.2 Power delivered to an AC circuit

Let's look again at our prototypical driven RLC circuit:



Assume that the EMF supplied is $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$. In the complex representation, this becomes $\tilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t}$.

We can now solve for the complex current in this circuit:

$$\begin{aligned}
\tilde{\mathcal{E}} &= \tilde{V}_R + \tilde{V}_L + \tilde{V}_C \\
&= \tilde{I}\left[R + i\omega L + \frac{1}{i\omega C}\right] \\
\longrightarrow \tilde{I} &= \frac{\tilde{\mathcal{E}}}{R + i[\omega L - 1/(\omega C)]} \,.
\end{aligned}$$

We now substitute $\tilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t}$, $\tilde{I}(t) = I_0 e^{-i\phi} e^{i\omega t}$. Then, we find

$$\begin{aligned}
I_0 &= \frac{\mathcal{E}_0}{\sqrt{R^2 + [\omega L - 1/(\omega C)]^2}} \\
&\equiv \mathcal{E}_0 / |Z_{\rm tot}(\omega)|
\end{aligned}$$

where $|Z_{\rm tot}(\omega)|$ is the magnitude of the total impedance. The phase is defined by

$$\begin{aligned}
\tan\phi &= -\frac{\mathrm{Im}[\tilde{I}]}{\mathrm{Re}[\tilde{I}]} \\
&= \frac{\omega L}{R} - \frac{1}{\omega C R} \,.
\end{aligned}$$

The *real* current flowing in this circuit is thus given by

$$I(t) = I_0 \cos(\omega t - \phi)$$

where

$$\begin{aligned} I_0 &= \frac{\mathcal{E}_0}{|Z_{\rm tot}(\omega)|} \\ \tan\phi &= \frac{\omega L}{R} - \frac{1}{\omega C R}\ . \end{aligned}$$

How much power is delivered to this circuit? From the definition of EMF as work per unit charge, and from the definition of current as charge per unit time, we know that the power delivered to this circuit must be

$$P(t) = I(t)\mathcal{E}(t)\ .$$

Let's plug in the results we have for $I(t)$ and $\mathcal{E}(t)$:

$$\begin{aligned} P(t) &= \frac{\mathcal{E}_0^2}{|Z_{\rm tot}(\omega)|}\cos\omega t\cos(\omega t - \phi) \\ &= \frac{\mathcal{E}_0^2}{|Z_{\rm tot}(\omega)|}\left[\cos\omega t\cos\omega t\cos\phi + \cos\omega t\sin\omega t\sin\phi\right]\ . \end{aligned}$$

On the second line, I have used a trig identity, $\cos(a-b) = \cos a\cos b + \sin a \sin b$, to expand the second cosine.

This result for $P(t)$ is exact. It tells us that the power delivered to the circuit is a quantity that oscillates. In many circumstances, the frequency of this oscillation is *far* too rapid for us to be able to track the detailed oscillations. For example, electricity out of the wall oscillates 60 times per second — way faster than anything we can keep track of without special equipment.

In this circumstance, when the oscillations are much faster than the phenomena that we are interested in, it makes a lot more sense to *average*. The "average" of any oscillating function is given by integrating that function over its period, and then dividing by the period. For example, the average power delivered to the circuit is given by

$$\begin{aligned} \langle P\rangle &= \frac{1}{T}\int_0^T P(t)\,dt \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P(t)\,dt\ . \end{aligned}$$

(On the last line, I'm using the fact that the period of a function whose angular frequency is ω is $2\pi/\omega$.)

The average power that the EMF delivers to the circuit is thus given by

$$\langle P\rangle = \frac{\mathcal{E}_0^2}{|Z_{\rm tot}(\omega)|}\frac{\omega}{2\pi}\left[\int_0^{2\pi/\omega}(\cos\omega t\cos\omega t\cos\phi)\ dt + \int_0^{2\pi/\omega}(\cos\omega t\sin\omega t\sin\phi)\ dt\right]\ .$$

The integrals we need are

$$\begin{aligned} \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(\cos\omega t\cos\omega t\cos\phi)\ dt &= \frac{1}{2}\cos\phi \\ \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(\cos\omega t\sin\omega t\cos\phi)\ dt &= 0\ . \end{aligned}$$

The average power delivered to the circuit is thus

$$\langle P(\omega)\rangle = \frac{1}{2}\frac{\mathcal{E}_0^2\cos\phi}{|Z_{\rm tot}(\omega)|}\ .$$

(I've written the power as a function of ω to emphasize its dependence on the driving frequency.) This average power is the one that is used to describe most common household items. For example, when we talk about a 100 Watt lightbulb, we mean that the average power delivered to the bulb is 100 Watts.

This power is often written in terms of the "root mean squared", or RMS, voltage. The RMS is often used when we talk about oscillating quantities whose average values are zero: we average the *square* of the quantity. For example, the RMS EMF is given by

$$\begin{aligned}
\mathcal{E}_{\rm rms}^2 &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\mathcal{E}(t)^2\,dt\\
&= \frac{\mathcal{E}_0^2}{2}\\
\longrightarrow \mathcal{E}_{\rm rms} &= \frac{\mathcal{E}_0}{\sqrt{2}}\ .
\end{aligned}$$

For purely sinusoidal AC EMF and current, the RMS value is just the amplitude divided by $\sqrt{2}$. Typically, when AC voltages are discussed, it is the RMS value that is indicated. For example, the wall voltage throughout North America is typically 120 Volts. This is the RMS value of the voltage from the wall; the *amplitude* of the voltage is $\sqrt{2}\times 120 = 170$ Volts. Note that, since $I_0 = \mathcal{E}_0/|Z_{\rm tot}|$, we also have $I_{\rm rms} = \mathcal{E}_{\rm rms}/|Z_{\rm tot}|$.

The average power delivered by the AC EMF can thus be written

$$\langle P(\omega)\rangle = \frac{\mathcal{E}_{\rm rms}^2\cos\phi}{|Z_{\rm tot}(\omega)|}\ .$$

This is the form in which we most commonly see the AC power related to AC voltage (or currents).

With a little bit of effort, we can show that

$$\cos\phi = \frac{R}{\sqrt{R^2+[\omega L - 1/(\omega C)]^2}} = \frac{R}{|Z_{\rm tot}(\omega)|}\ .$$

Using this, the average power can be rewritten as follows:

$$\begin{aligned}
\langle P(\omega)\rangle &= \frac{\mathcal{E}_{\rm rms}^2 R}{|Z_{\rm tot}(\omega)|^2}\\
&= I_{\rm rms}(\omega)^2\,R\ .
\end{aligned}$$

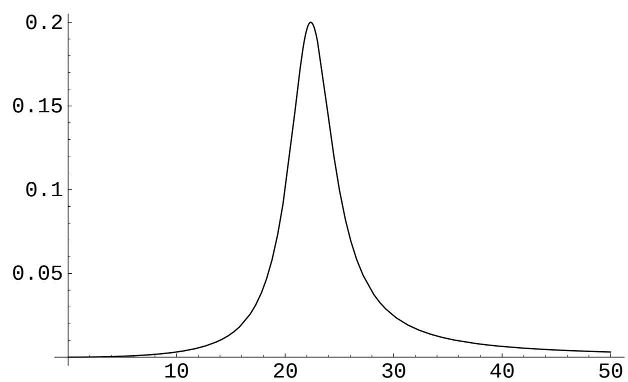
The average power delivered has a *really* simple form when expressed in terms of the RMS current — it looks just like the DC power dissipation formula! (Be a little careful, though: don't forget that the current in this case, $I_{\rm rms}$, is a function of ω.)

18.3 Resonance

The average power that is delivered to the circuit is a function of frequency. Using the formula

$$\langle P(\omega)\rangle = \frac{\mathcal{E}_{\rm rms}^2 R}{|Z_{\rm tot}(\omega)|^2}$$

we see that the frequency dependence enters through $1/|Z_{\rm tot}|^2$. This has the following appearance as a function of ω:



(The parameters $R = 5$ Ohms, $L = 1$ Henry, $C = 2 \times 10^{-3}$ Farads were used for this plot.) Clearly, there is some frequency at which the amount of power delivered is a maximum — at this frequency, the circuit is *in resonance* with the driving EMF. Our goal now will be to calculate this frequency: we look for maxima of

$$\frac{1}{|Z_{\rm tot}(\omega)|^2} = \frac{1}{R^2 + [\omega L - 1/(\omega C)]^2} .$$

We could do the usual procedure of looking for frequencies at which the derivative of this with respect to ω goes to zero; however, the answer is actually obvious by inspection! The maximum occurs when the ωL and $1/\omega C$ terms cancel each other out. This happens when

$$\omega = \omega_{\rm res} = \frac{1}{\sqrt{LC}} .$$

Notice that the resonant frequency is ω_0, the oscillation frequency of the pure LC circuit! At this frequency, $|Z_{\rm tot}| = R$, and so

$$\langle P(\omega_{\rm res})\rangle = \langle P_{\rm max}\rangle = \frac{\mathcal{E}_{\rm rms}^2}{R} .$$

Also at this frequency, we must have $\phi = 0$ — the current in the circuit is exactly in phase with the driving EMF. The voltage drops due to the capacitor and the inductor precisely cancel each other out — it is as though only the resistor matters in the circuit.

When one talks about a resonant system, one would like to quantify how "good" the resonance is. A "good" resonance is one in which the power versus ω curve is very narrow — the power delivered has a very sharp peak. A convenenient way to quantify this is to look for the two frequencies, $\omega_{\rm high}$ and $\omega_{\rm low}$, at which the power delivered has fallen to 1/2 of the peak value:

$$\langle P(\omega_{\rm high})\rangle = \langle P(\omega_{\rm low})\rangle = \frac{1}{2}\langle P_{\rm max}\rangle \; .$$

This tells us that

$$\begin{aligned}\frac{1}{2R} &= \frac{R}{|Z_{\rm tot}(\omega_{\rm high/low})|^2} \\ &= \frac{R}{R^2 + [\omega_{\rm high/low}L - 1/(\omega_{\rm high/low}C)]^2} \; .\end{aligned}$$

This in turn tells us that

$$\omega_{\rm high/low}L - \frac{1}{\omega_{\rm high/low}C} = R \; .$$

This equation can be rearranged to form a quadratic equation; solving it yields the two frequencies $\omega_{\rm high}$ and $\omega_{\rm low}$.

The "width" of the peak $\Delta\omega$ is then defined as $\Delta\omega = \omega_{\rm high} - \omega_{\rm low}$. The ratio of this width to the resonant frequency is then defined as the *quality factor* of the driven system:

$$Q \equiv \frac{\omega_{\rm res}}{\Delta\omega} \; .$$

On an upcoming problem set, you will show that limit,

$$\Delta\omega = \frac{R}{L} \; .$$

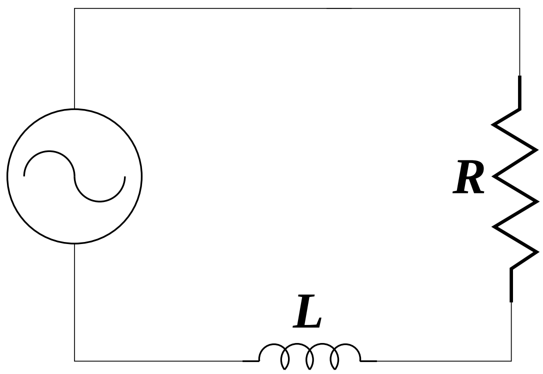
Combining this with the fact that $\omega_{\rm res} = \omega_0$ (the oscillation frequency of the LC circuit) and we have

$$Q = \frac{\omega_0 L}{R} \; .$$

This is *exactly* the same quality factor that we determined for the *un*-driven RLC circuit! This is not an accident. In the undriven system, the quality factor tells us about how quickly the system loses energy. In the driven system, it tells us about the power delivered to the system on resonance compared to slightly off resonance. Although it is beyond the scope of this course to prove this, these two concepts are intimately related to one another.

18.4 Filters

The frequency dependent response of an RLC circuit means that they can act as *filters* — different frequencies are preferentially "let through" different parts of the circuit. To see this principle in action, consider the following simple LR circuit with an AC driving EMF:



Working in the complex representation, we know that

$$\tilde{\mathcal{E}} = \tilde{V}_R + \tilde{V}_L \ .$$

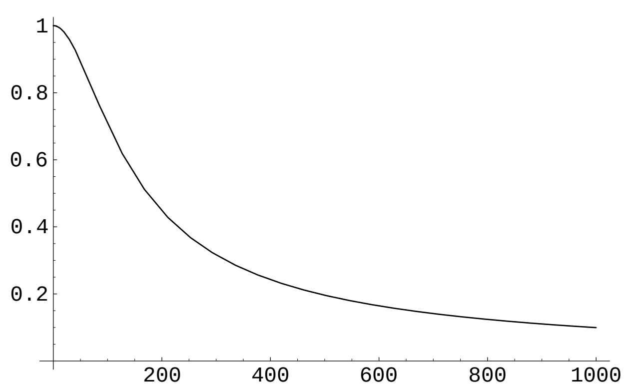
We also know there is a single current passing through both the resistor and the inductor:

$$\tilde{\mathcal{E}} = \tilde{I}\,(R + i\omega L)$$
$$\tilde{I} = \frac{\tilde{\mathcal{E}}}{R + i\omega L} \ .$$

What then is the *magnitude* of the voltage drop across the resistor? Multiplying by R and taking the magnitudes, we find

$$V_R = |\tilde{I}R| = \mathcal{E}\frac{R}{\sqrt{R^2 + \omega^2 L^2}} \ .$$

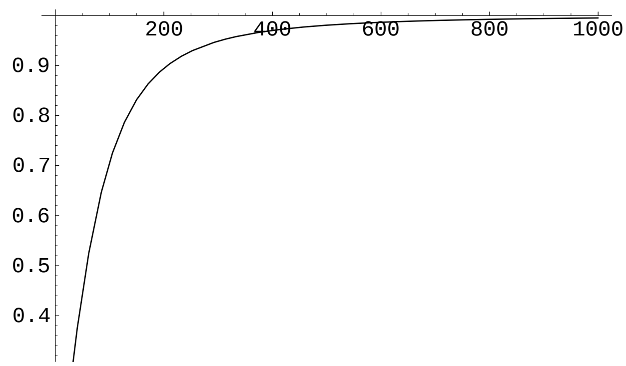
In this circuit, the resistor acts like a *low pass filter*: the voltage at low frequencies is essentially the same magnitude as the "input" voltage. However, at high frequencies, the voltage is very suppressed. Plotting V_R versus ω, we find something like this:



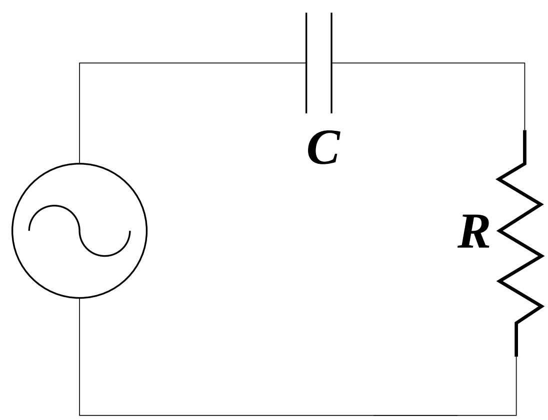
How about the magnitude of the voltage drop across the inductor? In this case, we have

$$V_L = |\tilde{I} Z_L| = \mathcal{E}\frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \; .$$

The inductor acts like a *high pass filter*: high frequency parts of the signal get through, but the stuff at low frequencies is highly suppressed: Plotting V_L versus ω gives us



How about if we have a circuit like this:



This again gives us filtering behavior. The complex current in the circuit is given by

$$\begin{aligned}\tilde{I} &= \frac{\tilde{\mathcal{E}}}{R + 1/(i\omega C)} \\ &= \frac{i\omega C\tilde{\mathcal{E}}}{1 + i\omega RC} \; .\end{aligned}$$

The voltage across the resistor is

$$V_R = |\tilde{I} R| = \mathcal{E}\frac{\omega C R}{\sqrt{1 + \omega^2 R^2 C^2}} \; .$$

This acts like a *high* pass filter: $V_R = 0$ at $\omega = 0$, but asymptotically approaches $\mathcal{E}$ as we go to high frequency.

The voltage across the capacitor is

$$V_C = |\tilde{I} Z_C| = \mathcal{E} \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \,.$$

This acts as a *low* pass filter.

The rule that capacitors act as low pass devices and inductors as high pass devices follows quite simply from the form of their impedance: A capacitor has a large impedance at low frequency,

$$Z_C = 1/(i\omega C)$$

and so shows a large voltage drop at low frequency. Likewise, an inductor has a large impedance at high frequency,

$$Z_L = i\omega L$$

and hence shows a large voltage drop at high frequency.