Scott Hughes *28 April 2005*

Massachusetts Institute of Technology
Department of Physics
8.022 Spring 2005

Lecture 20:
Wave equation & electromagnetic radiation

20.1 Revisiting Maxwell's equations

In our last lecture, we finally ended up with Maxwell's equations, the four equations which encapsulate everything we know about electricity and magnetism. These equations are:

$$\begin{aligned}
\text{Gauss's law:} &\qquad \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\
\text{Magnetic law:} &\qquad \vec{\nabla}\cdot\vec{B} = 0 \\
\text{Faraday's law:} &\qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \\
\text{Generalized Ampere's law:} &\qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\ .
\end{aligned}$$

In this lecture, we will focus on the source free versions of these equations: we set $\rho = 0$ and $\vec{J} = 0$. We then have

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 0 \\
\vec{\nabla}\cdot\vec{B} &= 0 \\
\vec{\nabla}\times\vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \\
\vec{\nabla}\times\vec{B} &= \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\ .
\end{aligned}$$

The source free Maxwell's equations show us that $\vec{E}$ and $\vec{B}$ are *coupled*: variations in $\vec{E}$ act as a source for $\vec{B}$, which in turn acts as a source for $\vec{E}$, which in turn acts as a source for $\vec{B}$, which ... The goal of this lecture is to fully understand this coupled behavior.

To do so, we will find it easiest to first *uncouple* these equations. We do this by taking the curl of each equation. Let's begin by looking at

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right)\ .$$

The curl of the left-hand side of this equation is

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}\left(\vec{\nabla}\cdot E\right) - \nabla^2\vec{E} = -\nabla^2\vec{E}\ .$$

(We used this curl identity back in Lecture 13; it is simple to prove, albeit not exactly something you'd want to do at a party.) The simplification follows because we have restricted ourselves to the source free equations — we have $\vec{\nabla}\cdot\vec{E} = 0$. Now, look at the curl of the right-hand side:

$$\vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\ .$$

Putting the left and right sides together, we end up with

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} = 0 \; .$$

Repeating this procedure for the other equation, we end up with something that is essentially identical, but for the magnetic field:

$$\frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \nabla^2 \vec{B} = 0 \; .$$

As we shall now discuss, these equations are particularly special and important.

20.2 The wave equation

Let us focus on the equation for $\vec{E}$; everything we do will obviously pertain to the $\vec{B}$ equation as well. Furthermore, we will simplify things initially by imagining that $\vec{E}$ only depends on x and t. The equation we derived for $\vec{E}$ then reduces to

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \frac{\partial^2 \vec{E}}{\partial x^2} = 0 \; .$$

20.2.1 General considerations

At this point, it is worth taking a brief detour to talk about equations of this form more generally. This equation for the electric field is a special case of

$$\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = 0 \; .$$

This equation is satisfied by *ANY* function whatsoever[1] provided that the argument of the function is written in the following special way:

$$f = f(x \pm vt) \; .$$

This is easy to prove. Let $u = x \pm vt$, so that $f = f(u)$. Then, using the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial f}{\partial u} = \frac{\partial f}{\partial u} \; ;$$

it follows quite obviously that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} \; .$$

The time derivatives are not quite so trivial:

$$\frac{\partial f}{\partial t} = \frac{\partial u}{\partial t} \frac{\partial f}{\partial u} = \pm v \frac{\partial f}{\partial u} \; ,$$

[1] To the mathematical purists, we must insist that the function f be smooth enough that it has well-behaved derivatives. At the 8.022 level, this is a detail we can largely ignore.

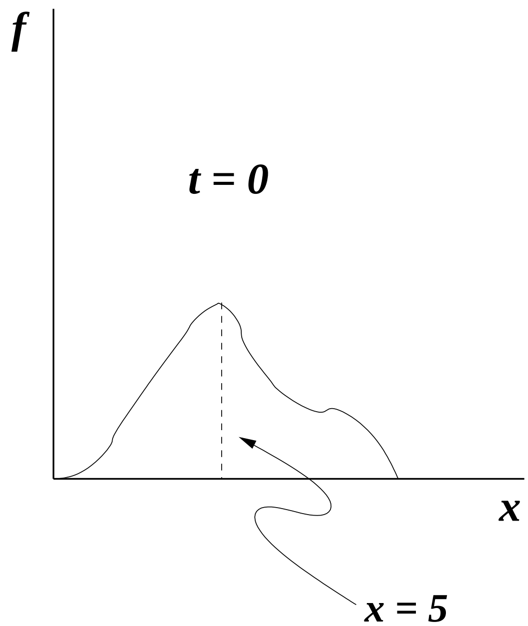
so

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial u^2} \ .$$

Plugging in, we see

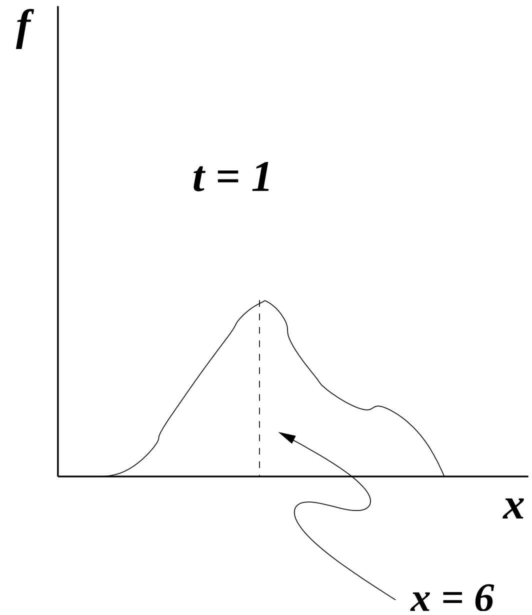
$$\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = v^2 \frac{\partial^2 f}{\partial u^2} - v^2 \frac{\partial^2 f}{\partial u^2} = 0 \ .$$

Having shown that any function of the form $f(x \pm vt)$ satisfies this equation, we now need to examine what this solution means. Let's consider $f(x - vt)$ first; and, let's say that $v = 1$. (Don't worry about units just yet.) Suppose that at $t = 0$, the function f has the following shape:



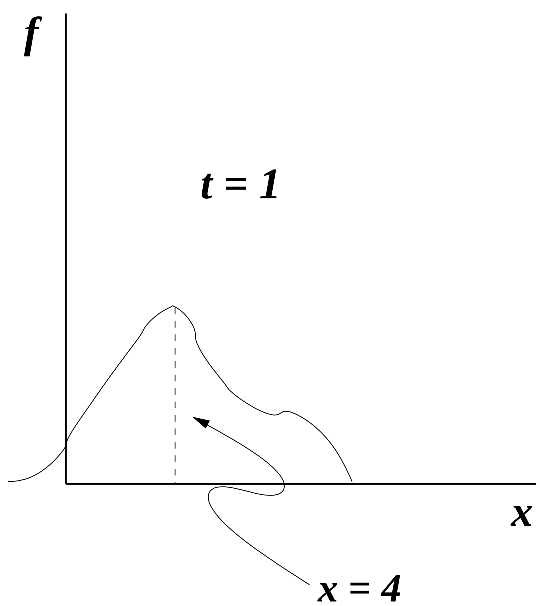
The peak of this function occurs when the argument of f is 5 — in other words, for $t = 0$, the peak occurs at $x = 5$.

What does this function look at $t = 1$? Clearly it will have the exact same shape. *However*, that shape will be shifted over on the x-axis by one unit! For example, the peak of f still occurs when the argument of f is 5. This means we must have $x - vt = 5$, which for $t = 1$ means the peak occurs at $x = 6$. All other points in the shape are similarly shifted:



As time passes, this shape just slides to the right, in the direction of increasing x. In other words, $f(x - vt)$ *representes a wave traveling in the POSITIVE x direction with speed v.*

What is the other solution? Applying this logic to $f(x + vt)$, we see that this represents a wave moving in the opposite direction, towards smaller x. For example, at $t = 1$, the $x + vt = 5$ means $x = 4$ — the peak of the curve shifts to $x = 4$ after 1 time unit:



$f(x + vt)$ *representes a wave traveling in the NEGATIVE x direction with speed v.*

OK, what we've got so far is that equations of the form

$$\frac{\partial^2 f}{\partial t^2} - v^2 \frac{\partial^2 f}{\partial x^2} = 0$$

generically have solutions of the form $f = f(x \pm vt)$, where $f(u)$ is essentially arbitrary. These solutions represent waves travelling at speed v; the solution with $x - vt$ represents a wave travelling in the positive x direction; the solution with $x + vt$ represents a wave travelling in the negative x direction.

20.2.2 Maxwell and the wave equation

You can probably see where we're going with this now! The equation for $\vec{E}$ that we derived earlier,

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \frac{\partial^2 \vec{E}}{\partial x^2} = 0$$

is itself a wave equation. It represents a wave travelling along the x axis with speed c — the speed of light! This kind of analysis is what made people realize that light is itself an electromagnetic wave.

This would have been far more interesting if we had worked in SI units throughout — getting the speed of light at the end of the day in cgs units is one of those "Well, duh" moments, since c appears throughout those equations already. In SI units, the wave equation that we end up with is

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\epsilon_0 \mu_0} \frac{\partial^2 \vec{E}}{\partial x^2} = 0$$

indicating that electromagnetic waves travel with speed

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \ .$$

Plug in some numbers:

$$\begin{aligned} \epsilon_0 &= 8.85418 \times 10^{-12}\,\text{Coulomb}^2\ \text{Newton}^{-1}\ \text{meter}^{-2} \\ \mu_0 &= 4\pi \times 10^{-7}\,\text{Newton sec}^2\ \text{Coulomb}^{-2} \end{aligned}$$

so,

$$\begin{aligned} \epsilon_0 \mu_0 &= 1.11265 \times 10^{-17}\,\text{s}^2\ \text{m}^{-2} \\ \longrightarrow \frac{1}{\sqrt{\epsilon_0 \mu_0}} &= 2.998 \times 10^{8}\,\text{m/s} = 2.998 \times 10^{10}\,\text{cm/s} \ . \end{aligned}$$

This again is of course the speed of light.

The fact that light is connected to electric and magnetic fields was first recognized from this kind of calculation. Maxwell first showed that the four equations describing electric and magnetic fields could be rearranged to form a wave equation, with a speed of propagation as given above.

At the time, ϵ_0 and μ_0 were empirical quantities that were determined through careful measurements of static electric and magnetic fields — in SI units, ϵ_0 shows up in the capacitance formula, and μ_0 shows up in the inductance formula . It was quite a shock to find that these quantities determined the behavior of highly *dynamical* fields as well, and even more of a shock to realize that the speed $1/\sqrt{\epsilon_0\mu_0}$ was the speed of light! At the time, ϵ_0, μ_0, and c were uncertain enough that the correspondance between $1/\sqrt{\epsilon_0\mu_0}$ was not perfectly definitive. However, it was close enough that Maxwell wrote in 1864[2]

This velocity is so nearly that of light that it seems we have strong reason to conclude that light itself (including radiant heat and other radiations) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.

It is largely for this reason that this whole set of equations is named for Maxwell.

20.3 Plane waves

A particularly important and useful form of wave for our study are the class known as *plane waves.* The most general form of these waves is

$$\begin{aligned}\vec{E} &= \vec{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)\\ &= \vec{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)\ ,\\ \vec{B} &= \vec{B}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)\\ &= \vec{B}_0 \sin(k_x x + k_y y + k_z z - \omega t)\ .\end{aligned}$$

The quantities $\vec{E}_0$ and $\vec{B}_0$ are the amplitudes of the electric and magnetic fields. Note that these amplitudes are vectors; we will determine some important properties for these vectors soon.

The vector $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ is the "wavevector"; its magnitude $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ is the "wavenumber". The unit vector $\hat{k} = \vec{k}/k$ is the wave's propagation direction. The number ω is the angular frequency of the wave.

Earlier, we showed that electromagnetic waves are given by functions of the form $f(x-ct)$. Generalizing to three dimensions, we would write this as $f(\vec{r}-c\hat{k}t)$ — the position $\vec{r}$ changes with velocity $c\hat{k}$. With this in mind, we can quickly show that the wavenumber k and ω are related to one another:

$$\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot\left(\vec{r} - \frac{\omega}{k}\hat{k}t\right)\ .$$

Combining this with the requirement that the dependence on $\vec{r}$ and t must be of the form $f(\vec{r} - c\hat{k}t)$ tells us that k and ω are related by the speed of light:

$$\omega = ck\ .$$

[2]See R. Baierlein, Newton to Einstein (Cambridge, 1992).

20.3.1 More on $\vec{k}$ and ω

For the moment, let's simplify our wave vector so that it is oriented along the x-axis: we put $\vec{k} = k\hat{x}$. Our plane wave solution is then

$$\begin{aligned}\vec{E} &= \vec{E}_0 \sin(kx - \omega t) \\ \vec{B} &= \vec{B}_0 \sin(kx - \omega t) \ .\end{aligned}$$

Let's look at the spatial variation of this wave at $t = 0$: clearly, we just have

$$\begin{aligned}\vec{E} &= \vec{E}_0 \sin(kx) \\ \vec{B} &= \vec{B}_0 \sin(kx) \ .\end{aligned}$$

These functions pass through zero at $x = 0$, $x = 2\pi/k$, $x = 4\pi/k$, etc. $2\pi/k$ is clearly a special lengthscale: it is called the *wavelength*, λ. It tells us about the lengthscale over which the wave repeats its pattern.

Now, think about the time variations at a particular location, say $x = 0$:

$$\begin{aligned}\vec{E} &= -\vec{E}_0 \sin(\omega t) \\ \vec{B} &= -\vec{B}_0 \sin(\omega t) \ .\end{aligned}$$

These functions pass through zero at $t = 0$, $t = 2\pi/\omega$, $t = 4\pi/\omega$, etc. $2\pi/\omega$ is the *period* of the wave, T. This of course the usual relationship between the period of an oscillation and its angular frequency. The inverse period is the wave's frequency (as opposed to *angular* frequency), usually written ν or f:

$$\nu = f = \omega/2\pi = 1/T \ .$$

20.3.2 Summary of wavenumber and frequency relations

Putting all of this together we end up with several relations between wavenumber, wavelength, frequency, angular frequency, and the speed of light. First, we have

$$\omega = ck \ .$$

Plugging in $\omega = 2\pi\nu$ and $k = 2\pi/\lambda$, we find

$$\nu\lambda = c \ .$$

20.4 Final constraints

Let's assess what we have so far: the source free Maxwell equations,

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \ ,\end{aligned}$$

can be manipulated in such a way as to show that $\vec{E}$ and $\vec{B}$ each obey the wave equation,

$$\begin{aligned}
\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} &= 0 \\
\frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \nabla^2 \vec{B} &= 0 \,.
\end{aligned}$$

We're not done yet! In addition to satisfying the wave equation, we must also verify that our functions $\vec{E}$ and $\vec{B}$ satisfy the *other* Maxwell equations — I can write down *lots* of functions that satisfy the wave equations, but that are not valid electromagnetic fields for $\rho = 0$ and $\vec{J} = 0$.

First, $\vec{E}$ and $\vec{B}$ must satisfy $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$. Let's plug our plane wave solutions into this:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \vec{\nabla} \cdot \left[\vec{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)\right] \\
&= (k_x E_{0,x} + k_y E_{0,y} + k_z E_{0,z}) \cos(k_x x + k_y y + k_z z - \omega t) \\
&= \left(\vec{k} \cdot \vec{E}_0\right) \cos(k_x x + k_y y + k_z z - \omega t) \,.
\end{aligned}$$

Likewise,

$$\vec{\nabla} \cdot \vec{B} = \left(\vec{k} \cdot \vec{B}_0\right) \cos(k_x x + k_y y + k_z z - \omega t) \,.$$

The only way for this to hold for *all* x and t is if

$$\begin{aligned}
\vec{k} \cdot \vec{E}_0 &= 0 \\
\vec{k} \cdot \vec{B}_0 &= 0 \,.
\end{aligned}$$

In other words, the divergence equations require that

Radiation's electric and magnetic fields are orthogonal to the propagation direction.

Let's next look at one of the curl equations:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \,.$$

This equation tells us that $\vec{E}$ and $\vec{B}$ are perpendicular to one another. On the right-hand side, we have $\partial \vec{E}/\partial t$. This points in the same direction as $\vec{E}$ — the time derivative does not change the vector's direction. On the left, we have $\vec{\nabla} \times \vec{B}$. This is perpendicular to $\vec{B}$ — the curl of any vector is orthogonal to that vector.

So, the first thing the curl equations tell us is that

Radiation's electric and magnetic fields are orthogonal to each other.

Now, let's actually compute the derivatives. The right-hand side is easiest, so let's do it first:

$$\begin{aligned}
\frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= -\frac{\omega}{c} \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \\
&= -k \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \,.
\end{aligned}$$

Now, the left-hand side:

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \vec{\nabla} \times \left[\vec{B}_0 \sin\left(\vec{k}\cdot\vec{r} - \omega t\right)\right] \\ &= \vec{\nabla} \times \left[\vec{B}_0 \sin\left(k_x x + k_y y + k_z z - \omega t\right)\right] \ .\end{aligned}$$

This looks like it might be a mess. However, back on Pset 3 we proved a very useful identity:

$$\vec{\nabla} \times (\vec{F} f) = f \vec{\nabla} \times \vec{F} + \vec{\nabla} f \times \vec{F} \ .$$

Put $\vec{F} \equiv \vec{B}_0$, $f = \sin\left(\vec{k}\cdot\vec{r} - \omega t\right)$. Then, we have $\vec{\nabla} \times \vec{F} = 0$ — $\vec{B}_0$ is constant! Only the other term contributes; plugging in for $\vec{F}$ and f, what we are left with is

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \vec{\nabla}\left[\sin\left(k_x x + k_y y + k_z z - \omega t\right)\right] \times \vec{B}_0 \\ &= \left[(k_x \hat{x} + k_y \hat{y} + k_z \hat{z}) \cos\left(k_x x + k_y y + k_z z - \omega t\right)\right] \times \vec{B}_0 \\ &= \left(\vec{k} \times \vec{B}_0\right) \cos\left(\vec{k}\cdot\vec{r} - \omega t\right) \ .\end{aligned}$$

After slogging through all of this, we finally obtain

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \\ \longrightarrow \left(\vec{k} \times \vec{B}_0\right) \cos\left(\vec{k}\cdot\vec{r} - \omega t\right) &= -k\vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t) \ ,\end{aligned}$$

or

$$\vec{E}_0 = -\hat{k} \times \vec{B}_0 \ .$$

There are two interesting things we can do with this. First, let's take the magnitude of both sides:

$$\begin{aligned}|\vec{E}_0| &= |-\hat{k} \times \vec{B}_0| \\ &= |\hat{k}||\vec{B}_0| \qquad \text{since } \hat{k} \text{ is orthogonal to } \vec{B}_0 \\ &= |\vec{B}_0| \qquad \text{since } |\hat{k}| = 1.\end{aligned}$$

In cgs units, the radiation's electric and magnetic fields have the same magnitude.

Second, one can show from this equation that $\vec{E}_0 \times \vec{B}_0 = |\vec{E}_0|^2 \hat{k}$:

$$\begin{aligned}\vec{E}_0 \times \vec{B}_0 &= -(\hat{k} \times \vec{B}_0) \times \vec{B}_0 \\ &= \vec{B}_0 \times (\hat{k} \times \vec{B}_0) \\ &= \hat{k}\left(\vec{B}_0 \cdot \vec{B}_0\right) - \vec{B}_0\left(\hat{k} \cdot \vec{B}_0\right) \\ &= \hat{k}|\vec{E}_0|^2 \ .\end{aligned}$$

To go from the 1st line to the 2nd line, I used $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$; to go to the 3rd line I used the vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$. I won't prove this identity explicitly; it's simple (albeit fairly tedious) to prove this in Cartesian coordinates. Finally, to go to the last line I used the results we just proved about the magnitude of the $\vec{B}$ field and its orthogonality to $\hat{k}$. The punchline is that

$\vec{E}_0 \times \vec{B}_0$ **is parallel to the propagation direction** $\hat{k}$**.**

As we shall soon see, $\vec{E} \times \vec{B}$ has a separate, *very* important physical meaning: Up to a constant, $\vec{E} \times \vec{B}$ tells us about the flow of energy that is carried by the radiation. This makes it very clear that $\vec{E} \times \vec{B}$ must be parallel to $\hat{k}$!