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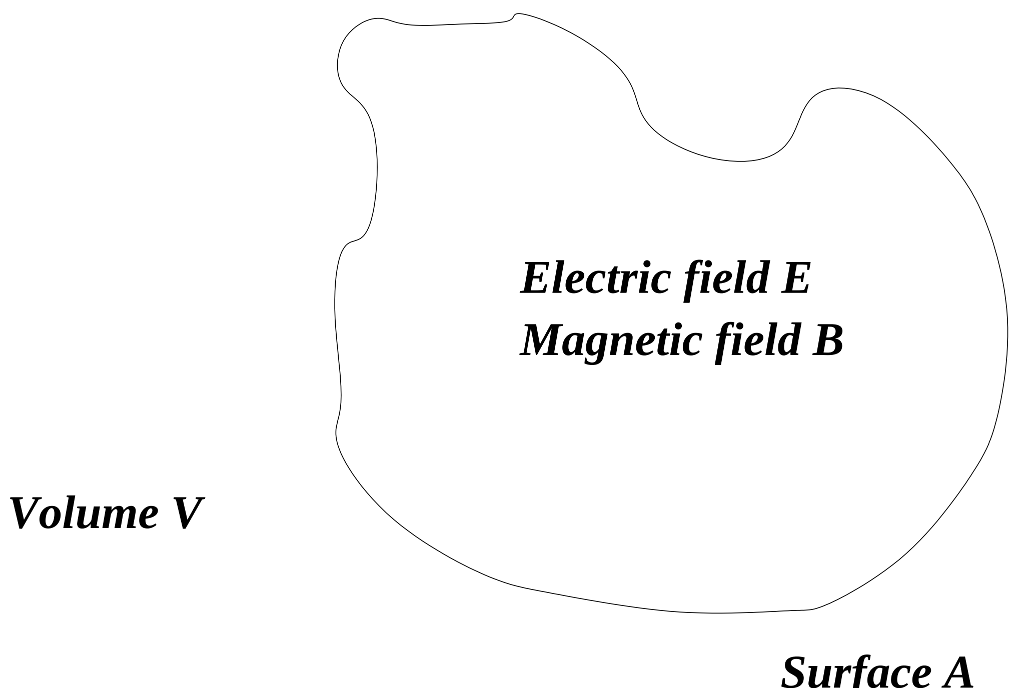
Lecture 22:
The Poynting vector: energy and momentum in radiation.
Transmission lines.

22.1 Electromagnetic energy

It is intuitively obvious that electromagnetic radiation carries energy — otherwise, the sun would do a pretty lousy job keeping the earth warm. In this lecture, we will work out how to describe the flow of energy carried by electromagnetic waves.

To begin, consider some volume V. Let its surface be the area A. This volume contains some mixture of electric and magnetic fields:



The electromagnetic energy density in this volume is given by

$$\frac{\text{energy}}{\text{volume}} = u = \frac{1}{8\pi}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) \ .$$

We are interested in understanding how the total energy,

$$U = \int_V u\,dV = \frac{1}{8\pi}\int_V dV\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right)$$

changes as a function of time. So, we take its derivative:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t}\frac{1}{8\pi}\int_V dV\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right) \ .$$

Since we have fixed the integration region, we can take the $\partial/\partial t$ ender the integral:

$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{1}{8\pi}\int_V dV \frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B}\right)\\ &= \frac{1}{4\pi}\int_V dV\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right)\ .\end{aligned}$$

Rearranging the source free Maxwell equations,

$$\begin{aligned}\frac{\partial\vec{E}}{\partial t} &= c\vec{\nabla}\times\vec{B}\\ \frac{\partial\vec{B}}{\partial t} &= -c\vec{\nabla}\times\vec{E}\end{aligned}$$

we can get rid of the $\vec{E}$ and $\vec{B}$ time derivatives:

$$\frac{\partial U}{\partial t} = \frac{c}{4\pi}\int_V dV\left[\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)-\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right]\ .$$

22.2 The Poynting vector

The expression for $\partial U/\partial t$ given above is as far as we can go without invoking a vector identity. With a little effort, you should be able to prove that

$$\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = -\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)+\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\ .$$

Comparing with our expression for $\partial U/\partial t$, we see that this expression simplifies things:

$$\begin{aligned}\frac{\partial U}{\partial t} &= -\frac{c}{4\pi}\int_V dV\,\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)\\ &\equiv -\int_V dV\,\vec{\nabla}\cdot\vec{S}\ .\end{aligned}$$

On the second line we have defined

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}\ ;$$

we'll discuss this vector in greater detail very soon.

Since we have a volume integral of a divergence, the obvious thing to do here is to apply Gauss's theorem. This changes the integral over V into an integral over the surface A:

$$\frac{\partial U}{\partial t} = -\int_A d\vec{a}\cdot\vec{S}\ .$$

In other words, **The rate of change of the electromagnetic energy in the volume V is given by the flux of $\vec{S}$ through V's surface.** Notice the minus sign; this is easily explained. Recall that $d\vec{a}$ points outward. If $\vec{S}$ and $d\vec{a}$ are parallel, energy is *leaving* the system, so U decreases. And vice versa.

$\vec{S}$ is called the Poynting[1] vector. It points[2] in the same direction as the electromagnetic energy flow. (This gives a somewhat more physical picture of the fact that $\vec{E}\times\vec{B}$ tells us the

[1]It's actually named after a person, John Henry Poynting. You've got to wonder if he was fated to work this quantity out.

[2]This is usually when a lesser lecturer would make a terrible pun about the direction in which the flow of energy "Poynts". I'll spare you.

propagation direction for electromagnetic radiation.) As noted above, the flux of $\vec{S}$ through some surface A gives the total *power* flowing through A:

$$\text{Power through } A = \int_A \vec{S} \cdot d\vec{a} \, .$$

22.2.1 Dimensional analysis

The units of the Poynting vector are

$$\begin{aligned} \text{Poynting} &\equiv \text{velocity} \times \text{electric field} \times \text{magnetic field} \\ &\equiv \text{velocity} \times \text{electric field}^2 \end{aligned}$$

Electric field squared gives us energy per unit volume, so

$$\begin{aligned} \text{Poynting} &\equiv \frac{\text{length}}{\text{time}} \times \frac{\text{energy}}{\text{length}^3} \\ &= \frac{\text{power}}{\text{area}} \, . \end{aligned}$$

This is just what we should have if the flux of $\vec{S}$ is to be the power through an area. In cgs units, $\vec{S}$ has the units of erg/(s-cm^2).

It's worth noting at this point that the magnitude of $\vec{S}$ is a quantity known as the intensity, I. A very intense source of radiation is something that emits lots of power into a very small area.

22.3 Examples

22.3.1 Plane wave

Let's look at a linearly polarized plane wave, $\vec{E} = E_0\hat{x}\sin(kz - \omega t)$, $\vec{B} = E_0\hat{y}\sin(kz - \omega t)$. The Poynting vector associated with this wave is

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} = \frac{c}{4\pi}E_0^2\sin^2(kz - \omega t)\hat{k} \, .$$

It's useful to compare this to the energy density in the wave:

$$\begin{aligned} u &= \frac{1}{8\pi}\left(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}\right) \\ &= \frac{1}{4\pi}E_0^2\sin^2(kz - \omega t) \end{aligned}$$

Comparing, we see that

$$\vec{S} = u\vec{c}$$

where $\vec{c} = c\hat{k}$ is the vectorial velocity of the radiation. This is very nicely in keeping with the idea that $\vec{S}$ tells us about the flow of energy! (Note the similarity to the relation between current density and the velocity of moving charges, $\vec{J} = \rho\vec{v}$. $\vec{S}$ is a flow of energy density in the same way that $\vec{J}$ is a flow of charge density.)

For many kinds of radiation that we will wish to study, k and ω are so large that $\sin^2(kz - \omega t)$ oscillates extremely quickly. (For example, the frequency of visible light —

mean wavelength roughly 500 nanometers — is about 6×10^{14} Hz.) It makes a lot more sense to take the average of this quantity, which amounts to replacing the $\sin^2$ with 1/2:

$$\langle\vec{S}\rangle = \frac{c\hat{k}}{8\pi}E_0^2\ .$$

The intensity (magnitude of $\vec{S}$) is most typically discussed in terms of the averaged Poynting vector:

$$I = |\langle\vec{S}\rangle| = \frac{c}{8\pi}E_0^2\ .$$

22.3.2 Spherical wavefronts

If we solve the wave equation in spherical coordinates, one important solution is given by

$$\begin{aligned}\vec{E} &= \frac{\omega^2 p}{c^2}\sin\theta\frac{\sin(kr-\omega t)}{r}\hat{\theta}\\ \vec{B} &= \frac{\omega^2 p}{c^2}\sin\theta\frac{\sin(kr-\omega t)}{r}\hat{\phi}\ .\end{aligned}$$

(This solution is actually only good for distances $r \gg \lambda$, where $\lambda = 2\pi/k$ is the usual wavelength. This is a subtlety far beyond the scope of 8.022! For now, just trust me.)

This radiation solution is what we find for an oscillating dipole arranged so that the dipole vector, $\vec{p}$, lies parallel to the z axis (p is the magnitude of $\vec{p}$). Notice that the radiation propagates out *radially*: although there is some angular dependence, the radiation flows out through spheres.

The Poynting vector we construct for this wave is

$$\begin{aligned}\vec{S} &= \frac{c}{4\pi}\vec{E}\times\vec{B}\\ &= \frac{1}{4\pi c^3}\omega^4 p^2\sin^2\theta\frac{\sin^2(kr-\omega t)}{r^2}\hat{r}\ .\end{aligned}$$

Averaging over time, we have

$$\langle\vec{S}\rangle = \frac{\omega^4 p^2}{8\pi c^3 r^2}\sin^2\theta\hat{r}\ .$$

The Poynting vector (and hence the intensity) fall of with a $1/r^2$ law. This is extremely important! Suppose we draw a sphere of radius R around this system, centered on the origin (where the oscillating dipole is located). Let's now compute the total rate of energy flowing through this sphere:

$$\left\langle\frac{\partial U}{\partial t}\right\rangle = \int_R d\vec{a}\cdot\langle\vec{S}\rangle\ .$$

The area element on this sphere is $d\vec{a} = R^2\sin\theta\, d\theta\, d\phi\,\hat{r}$; the integral becomes

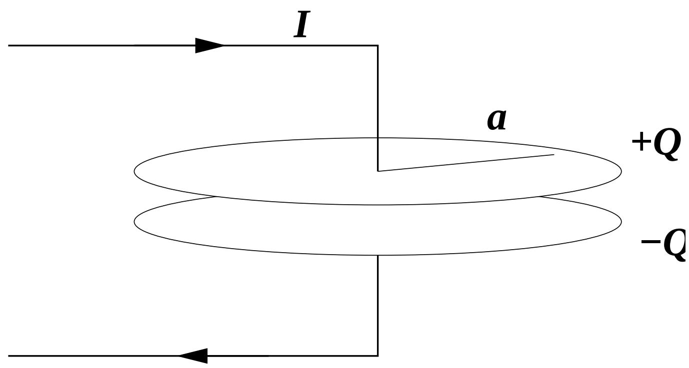
$$\begin{aligned}\left\langle\frac{\partial U}{\partial t}\right\rangle &= \int_0^{2\pi}d\phi\int_0^{\pi}d\theta\left(R^2\sin\theta\hat{r}\right)\cdot\left(\frac{\omega^4 p^2}{8\pi c^3 R^2}\sin^2\theta\hat{r}\right)\\ &= \frac{\omega^4 p^2}{8\pi c^3}\int_0^{2\pi}d\phi\int_0^{\pi}\sin^3\theta\, d\theta\\ &= \frac{\omega^4 p^2}{4c^3}\int_0^{\pi}\sin^3\theta\, d\theta\\ &= \frac{\omega^4 p^2}{3c^3}\ .\end{aligned}$$

This final result is a special case of what is known as the *Larmor formula*. Notice that this result is *completely independent of the radius* R! What happened is that the Poynting vector or intensity drops off like $1/r^2$. However, the total area of a spherical surface grows with r^2, so the *total rate of energy flow* is constant.

This, in essence, is telling us that electromagnetic waves conserve energy. They are spreading out over large areas, but the total energy they carry remains constant.

22.3.3 Capacitor

The Poynting vector applies in *any* situation in which both an electric and a magnetic field appear, not just with radiation. In this example, we will apply it to the charging up of a capacitor. Consider a capacitor that currently has a charge separation of Q, and has a current I flowing onto it:



The electric field between the plates is

$$E = 4\pi Q/A = 4Q/a^2 \ ;$$

it points down. From the generalized Ampere's law, we have

$$\oint \vec{B} \cdot d\vec{s} = \frac{1}{c}\frac{d\Phi_E}{dt}$$

We take the contour integral around a ring of radius r; the flux is computed through this ring:

$$\begin{aligned} 2\pi r B(r) &= \frac{\pi r^2}{c}\frac{dE}{dt} \\ &= \frac{\pi r^2}{c}\frac{4}{a^2}\frac{dQ}{dt} \\ &= \frac{4\pi I r^2}{ca^2} \ . \end{aligned}$$

So the magnetic field at radius r from the center of the capacitor is

$$B(r) = \frac{2Ir}{ca^2} \ .$$

The direction is of course circulational, defined by right hand rule with your thumb pointing down. This means that at the front of the capacitor $\vec{B}$ points to the left; at the back, it points to the right; etc.

The cool thing is to now evaluate the Poynting vector: since $\vec{E}$ and $\vec{B}$ are orthogonal, we clearly have

$$S(r) = \frac{c}{4\pi} EB = \frac{2QI}{\pi a^3} \ .$$

What is the direction? At the front, we have (down)×(left). This points *into* the capacitor. At the back, we have (down)×(right). This again points *into* the capacitor. In fact, we can quickly show that at *all* points, $\vec{S}$ points into the capacitor.

This makes perfect sense! It *should* point inwards since, as we well know, the amount of energy in the capacitor increases as the plates charge up. The Poynting vector provides us with a way of visualizing this.

You will explore this facet of the Poynting vector further on Pset 11.

22.4 Momentum carried by radiation

Since electromagnetic radiation carries energy, it won't surprise you to learn that it carries momentum as well. A simple way to see this is to make use of the relativistic mass-energy-momentum formula you worked out in Pset 6:

$$E^2 = |\vec{p}|^2 c^2 + m^2 c^4 \ .$$

If we plug in $m = 0$, we find

$$E = |\vec{p}|c \ ; \qquad |\vec{p}| = E/c \ .$$

Let's think about using the relationship with what we've worked out so far. The Poynting vector tells us about the rate at energy is delivered to a unit area, just as we learned in the dimensional analysis discussion above. If we divide by c, then what we must have is the rate at which *momentum* is delivered to a unit area:

$$\frac{\vec{S}}{c} = \frac{\vec{E} \times \vec{B}}{4\pi}$$

Dimensionally,

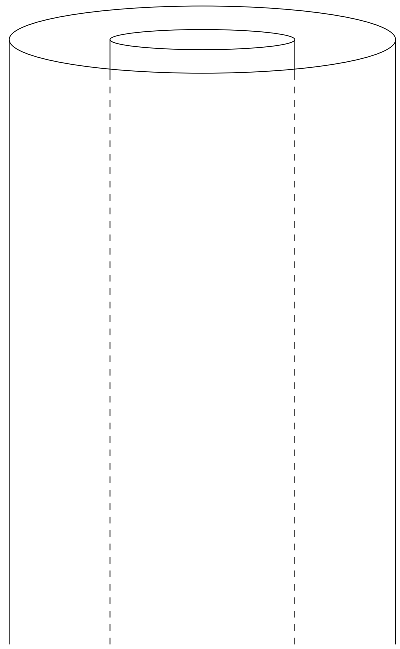
$$\frac{\vec{S}}{c} = \frac{\text{momentum}}{\text{time} \times \text{area}} = \frac{\text{force}}{\text{area}} = \text{pressure} \ .$$

Radiation exerts pressure! Intensity divided by the speed of light tells us how much pressure is exerted by some source of light.

22.5 Transmission lines

Suppose we want to send a signal from one device to another — our goal is to get electromagnetic energy from point A to point B. What's the best way to do this? If we just use wires, we're going to have problems — the wires will act like antennas and we'll lose a lot of energy to radiation. Instead, we would like to set up some kind of structure that *shields* the signal, so that no field leaks outside. This setup is called a transmission line.

The simplest example of a transmission line is a coaxial cable: a pair of conducting tubes nested in one another. Equal and opposite currents flow on the inner and the outer tubes,



so that the net current "seen" outside the cylinder is zero. Note that this is *not* a simple DC current — it is a highly oscillatory AC current. This means that there must be some charge distribution as well. Equal and opposite charges "live" on the inner and outer tubes of the coax.

Suppose that the inductance per unit length of this cable is L', and that its capacitance per unit length is C'. Consider a short length Δx of this cable. This little segment has a total inductance of $L'\Delta x$. If the current flowing down the cable is I and has the rate of change $\partial I/\partial t$, then the voltage drop due to inductance over this short length is

$$\Delta V = V(x+\Delta x) - V(x) = -(L'\Delta x)\frac{\partial I}{\partial t} \ .$$

Divide both sides by Δx and take the limit:

$$\frac{\partial V}{\partial x} = -L'\frac{\partial I}{\partial t} \ .$$

Consider now the charge on this little segment: since it is at potential $V(x)$, and its capacitance is $C'\Delta x$, the amount of charge on this segment is

$$\Delta Q = (C'\Delta x)V(x) \ .$$

Divide and take the limit:

$$\frac{\partial Q}{\partial x} = C'V(x) \ .$$

Let's take another derivative of this:

$$\begin{aligned}\frac{\partial^2 Q}{\partial x^2} &= C'\frac{\partial V}{\partial x} \\ &= -C'L'\frac{\partial I}{\partial t} \\ &= C'L'\frac{\partial^2 Q}{\partial t^2} \ .\end{aligned}$$

In going from the first to the second line, we used our previous result relating $\partial V/\partial x$ and $\partial I/\partial t$; in going from the second to the third line, we used $I = -\partial Q/\partial t$ (discharging capacitor current).

Rearranging this, we see yet another wave equation!

$$\frac{\partial^2 Q}{\partial t^2} - \frac{1}{C'L'}\frac{\partial^2 Q}{\partial x^2} = 0 \ .$$

With a little bit of tweaking, you should be able to convince yourself that this equation holds replacing Q with V and I as well. The speed of propagation of this wave is

$$v = \frac{1}{\sqrt{L'C'}} \ .$$

Now, for the coaxial cable introduced above, you should be able to show very easily that

$$\begin{aligned} C' &= \frac{1}{2\ln(b/a)} \\ L' &= \frac{2\ln(b/a)}{c^2} \ . \end{aligned}$$

(If you can't work it out, go back to old psets! You worked out the capacitance in Pset 4 and the inductance in Pset 8; you just to divide out the length l to put things on a "per unit length" basis.) This means that

$$v = \frac{1}{\sqrt{L'C'}} = c \ .$$

The wave propagates down the transmission line at the speed of light! Not surprising, at least in retrospect.

One other very important quantity determines the characteristic of a transmission line: its impedance. Following our intuition from our study of AC circuits, we define the impedance as the ratio of the voltage to the current:

$$Z = \frac{V}{I} = \frac{V}{\Delta Q/\Delta t} \ .$$

As we already discussed, for a small segment of the cable $\Delta Q = (C'\Delta x)V$, so

$$\frac{\Delta Q}{\Delta t} = C'V\frac{\Delta x}{\Delta t} = \frac{C'V}{\sqrt{L'C'}} = V\sqrt{\frac{C'}{L'}}$$

where we have used the fact that $\Delta x/\Delta t$ is just the speed of propagation down the line.

Putting everything together, we find

$$Z = \sqrt{\frac{L'}{C'}} \ .$$

For the coaxial cable, this yields

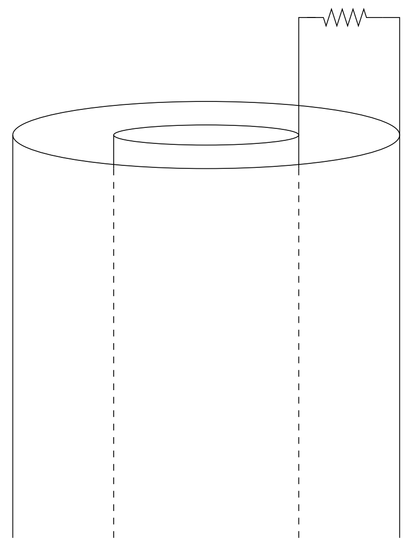
$$Z = \frac{2\ln(b/a)}{c} \ .$$

Using the conversion factor 1 Ohm $= 1.1 \times 10^{-12}$ sec/cm, this means

$$Z = 60\,\text{Ohms}\,\ln(b/a)\ .$$

Many coaxial cables are actually filled with a dielectric material (which we have not discussed in detail); this has the effect of increasing the capacitance, which reduces the line's characteristic impedance. 50 Ohms is a value that is commonly encountered (e.g., in cable TV systems).

One of the most important consequences of the impedance of a transmission line is in determining how it must be *terminated.* Roughly speaking, a terminated cable has the form



The resistor might actually be some electronic device, like a television or a computer; then again, it might just be a resistor. The point is that the signal which is flowing down the cable is then fed into this device or resistor.

If the resistance of this "load" precisely matches the impedance of the cable, then we have *matched* the impedances. Because V/I does not change in going from the cable to the resistor, it is as though the cable were of infinite length! There is no way for the signal to "know" whether it is in the resistor or in the cable.

Suppose the resistance were ∞ Ohms — an open circuit. Then, the currents would essentially bounce off the "resistor" and head back where they came from. The signal would just reverse course — you'd get a horrible reflection back up the line. Suppose the resistance were 0 Ohms — a short circuit. Then, the currents would just flow from the inner tube to the outer tube, and vice versa — the signal would reverse course *and* switch sign on its amplitude!

In general, if the impedance of the cable doesn't match the impedance of the load, you get reflections of this sort. This is one reason why cheap cable TV equipment can give you a horrible picture — there are these extra signals bouncing around, which eventually arrive at the TV late, and possible with a 180° phase shift. This causes ghost images to appear, and generally confuses your equipment.