MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
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LECTURE 3
FROM GALILEAN RELATIVITY TO ... ?

## 3.1 Speed of light is $c$ to all observers

Much of what we will study in the next several weeks is essentially an examination of the consequences of Einstein's hypothesis that all observers measure the speed of light to be $c$. The fact that all observers measure measure $c$ for the speed of light means that this quantity is an *invariant* — it is the same in all frames of reference. As you will hopefully come to appreciate over the course of this semester, invariants are incredibly useful: we can exploit the fact that they are the same for all observers to facilitate many of the calculations we will want to perform.

The invariance of the speed of light tells us that the amount of light's displacement per unit time is the same to all observers. In the Galilean transformation, we saw that displacement varies depending on frame; as a consequence speed (the magnitude of velocity) must vary as well. The Galilean transformation is thus inconsistent with the idea that the speed of light is the same to all observers; it must be corrected. If displacement varies according to the frame of an observer, but something's speed is invariant, we must find that *time intervals* vary by frame. Allowing the time interval to vary by frame is the only way that speed (displacement interval per unit time interval) can be invariant.

It's worth keeping in mind, however, that the Galilean transformation works very well in many circumstances, so it must be *approximately* correct. Our "generalized" transformation law had best be consistent with Galileo in some appropriate limit.

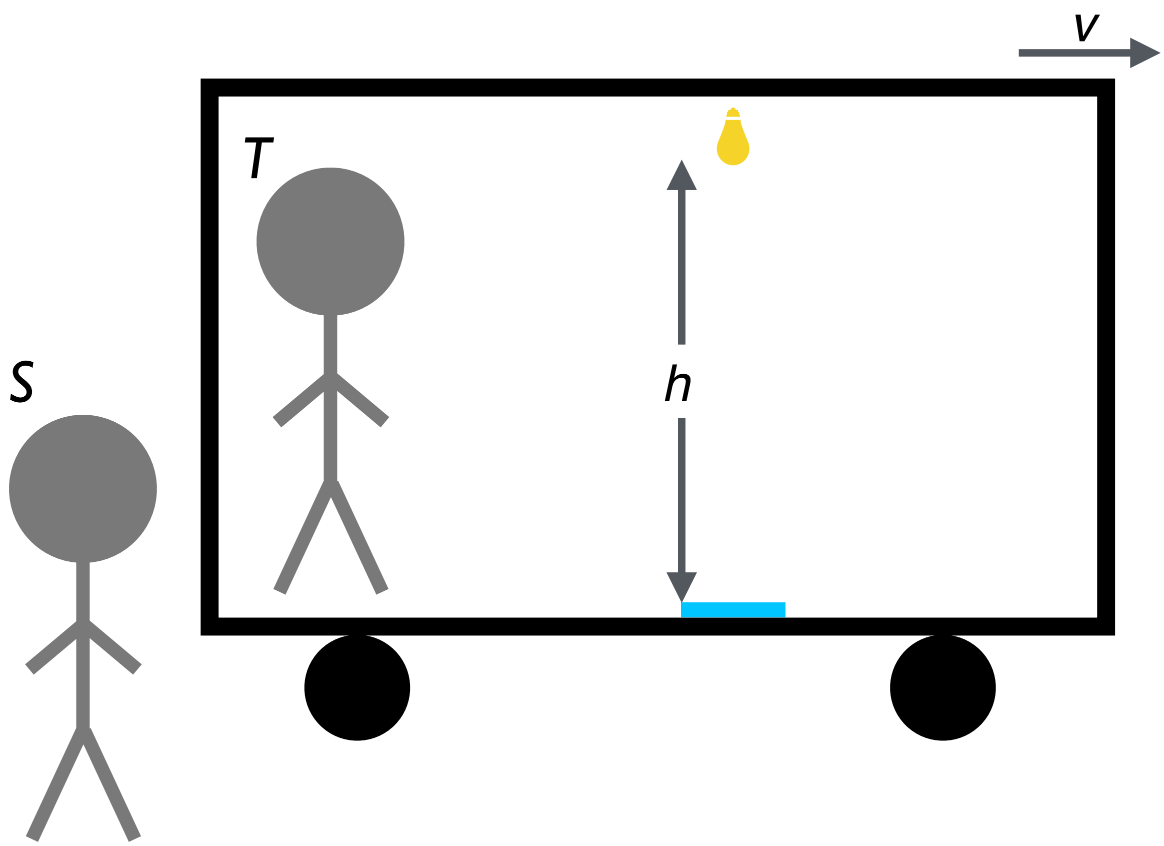
⌈ASIDE: It's worth noting that the invariance of the speed of light also means that it is a great thing on which to base a metrology standard. That's why we take $c$ to be *exactly* $2.99792458 \times 10^8$ meters per second. We then determine the meter to be the distance light travels in $1/(2.99792458 \times 10^8)$ seconds. Technique in atomic physics have taught us how to measure time intervals very precisely, so this is a way of getting the meter out that capitalizes on what we measure best.⌋

## 3.2 Consequences I

Before generalizing the Galilean transformation, let's work through a few "thought experiments" which illustrate some of the consequences of light speed's invariance. We will consider two observers: Observer $S$ is standing in a station; observer $T$ is standing in a train that is moving with speed $v$ through the station. These two observers each make measurements whose values we will compare.

First, imagine there is a light bulb inside the train. This bulb emits a pulse of light at some moment; we call this event $A$. The pulse of light propagates downward through the train, striking a photodetector on the floor, which records the moment the light strikes. We call this event $B$. Events $A$ and $B$ are geometric objects; all observers agree on the existence of these two things happening, though they may label the coordinates in time and space of these events differently.

We being our analysis by asking: What interval of time do observers $T$ and $S$ measure between events $A$ and $B$?




Let's do this first in observer $T$'s frame of reference. Observer $T$ sees the light move through a vertical displacement $h$, so they deduce

$$\Delta t_T = h/c\,. \tag{3.1}$$

Observers in the station agree that the light moves through a vertical distance $h$, but also see it move through a horizontal distance that depends on the train's speed:

$$\Delta t_S = D/c = \frac{\sqrt{h^2 + (v\Delta t_S)^2}}{c}\,, \tag{3.2}$$

from which we find

$$\Delta t_S = \frac{h/c}{\sqrt{1 - v^2/c^2}} \equiv \gamma \Delta t_T\,, \tag{3.3}$$

$$\text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \left(1 - \beta^2\right)^{-1/2}\,. \tag{3.4}$$

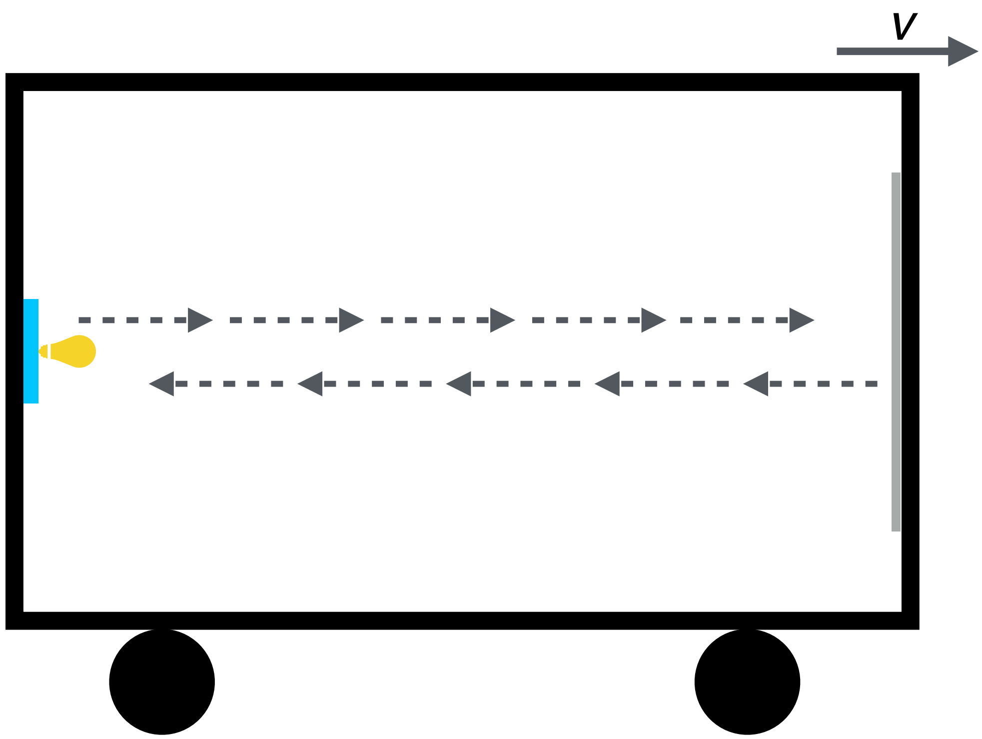
Notice that the factor $\gamma \geq 1$. The interval of time as measured on the station is *longer* than the interval measured on the train. For example, if the train moves at $v = \sqrt{3}c/2$ and the observers on the train measure 7 nanoseconds for the light to reach the photodetector, then observers in the station measure 14 nanoseconds for the light to reach the photodetector. Less time accumulates between the two events according to train observers than accumulates according to station observers.

*Moving clocks run slow.* This is a phenomenon known as *time dilation.*

⌈ASIDE: In doing this analysis, we've assumed that both the train and the station observers measure the same height $h$ for the light's vertical displacement. Hold that thought!⌋

## 3.3 Consequences II

Let's next imagine that we arrange the light pulse so that it travels to the front of the car, bounces off a mirror, and returns to a photosensor:




Both the train and station observers measure the time interval between the flash and the light striking the photodetector, and use this to infer the length of the train car. On the train (neglecting the size of the light bulb and the finite thickness of the sensor and mirror which are features of the sketch), observer measures a time interval of $\Delta t_T$ between the flash and the light striking the photodetector, and deduces that the train has a length

$$\Delta x_T = c\Delta t_T/2 \ . \tag{3.5}$$

The size measured by observers on the train is just the time it takes light to travel from one end of the train to the other and back, divided by two.

To compute the size measured by observers in the station, let's break the calculation into two pieces: one piece gives us the time to travel from the bulb flash to reach the mirror; the other times travel from the mirror back to the photodetector. The interval of time measured for these two legs is

$$\Delta t_{S,1} = \frac{(\Delta x_S + v\Delta t_{S,1})}{c} \quad \longrightarrow \quad \Delta t_{S,1} = \frac{\Delta x_S}{c - v} \; ; \tag{3.6}$$

$$\Delta t_{S,2} = \frac{(\Delta x_S - v\Delta t_{S,2})}{c} \quad \longrightarrow \quad \Delta t_{S,2} = \frac{\Delta x_S}{c + v} \; . \tag{3.7}$$

Notice the asymmetry[1] in the two contributions: the flash of light is "chasing" the mirror during interval 1, but is heading toward the advancing photodetector during interval 2. We add these two contributions to get the total travel time:

$$\begin{aligned} \Delta t_S = \Delta t_{S,1} + \Delta t_{S,2} &= \frac{2\Delta x_S/c}{1 - v^2/c^2} \\ &= 2\gamma^2 \Delta x_S/c \; . \end{aligned} \tag{3.8}$$

We have now related $\Delta t_S$ to $\Delta x_S$, and $\Delta t_T$ to $\Delta x_T$. What we really want is a relation between $\Delta x_S$ and $\Delta x_T$. To cut through the different relations, let's take advantage of our previous result that the moving clock runs slow, i.e. that $\Delta t_S = \gamma \Delta t_T$. Using this, we can rewrite Eq. (3.8) as

$$\gamma \Delta t_T = 2\gamma^2 \Delta x_S/c \; . \tag{3.9}$$

But we know that $\Delta t_T$ and $\Delta x_T$ are related by Eq. (3.5). Using this in Eq. (3.9) yields

$$\Delta x_S = \Delta x_T/\gamma \; . \tag{3.10}$$

This at last relates the spatial distance measured by the train observer to that measured by the station observer. Note that since $\gamma \geq 1$, Eq. (3.10) means the distance interval measured in the station is *shorter* than the distance interval measured on the train.

*Moving rulers are shortened along the direction of motion.* This is a phenomenon known as *length contraction*.

## 3.4 Consequences III

Are moving rulers affected along axes other than along the direction of motion? The answer is no: If they were, we would get *inconsistent physics* — different events occurring in different frames of reference.

Imagine a train going at a speed $v = \sqrt{3}c/2$, so that $\gamma = 2$. Suppose the train is 5 meters tall, and is approaching a tunnel whose opening is 8 meters high. If length contraction affected the train's height, we'd have a serious problem:

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[1] Some students have been bothered about why we are introducing this reflection. For now, I'm going to give the highly nonsatisfying answer that we *could* just do this with motion along one direction (i.e., not accounting for the bounce), but the details are "messier." By having the two events — emission of light; measurement of light — occur at the same spatial location in one of the two frames, it turns out there are some simplifications that occur "under the hood," so to speak. Once we have introduced and become comfortable with the Lorentz transformation, it will be far easier to do the "one way" version of this calculation. An analysis of this may appear on a future problem set...

- **Tunnel rest frame:** The train's height is contracted by a factor of $\gamma$, making it 2.5 meters tall — easily fitting into the 8 meter tunnel opening.
- **Train rest frame:** The tunnel's height is contracted by a factor of $\gamma$, making it 4 meters tall. The 5 meter train experiences a very high speed collision, destroying the train, the mountain into which the tunnel is carved, and a good fraction of the surrounding countryside.

We require all observers to agree on events, even if they describe them using different labels. But these two outcomes — train merrily passing through a tunnel in one frame; chaos, death, destruction, and sadness in another — are not mere differences of label. These are completely inconsistent outcomes.

*Moving rulers are unaffected along directions orthogonal to their direction of motion.* (Hence our assumption that both the train observer and the station observer measure a vertical displacement of $h$ was indeed correct.)

## 3.5 From Galileo to Lorentz

In the examples we've discussed above, we have allowed our notions of time and space intervals to get mixed up by our demand that all observers measure light to have a propagation speed of $c$. As we can see, this leads to some seemingly bizarre consequences. However, these consequences follow straightforwardly from our requirement that $c$ be an invariant. We are going to use invariants a lot in this course; they are your friends for the purpose of sorting through complicated calculations.

Let us now think about how to mix up different intervals in a more systematic manner. Galilean transformations allowed different inertial frames to define different standards for space: what's "left" to you is a mixture of "left" and "forward" to someone with a different orientation; what's "there" to you is "there and steadily moving farther away" to someone moving with a fixed speed. But time is the same for everyone.

Let's think about a category of transformations that can mix up space *and* time, doing so in such a way that the speed of light is left invariant. Let's think about a station observer who labels events with coordinates $(t_S, x_S, y_S, z_S)$, and a train observer who labels events with coordinates $(t_T, x_T, y_T, z_T)$. The station observer sees the train moving with $\mathbf{v} = v\mathbf{e}_x$.

We will begin by assuming that the train frame's coordinates are related to those of the station with the following linear relations:

$$t_T = At_S + Bx_S \tag{3.11}$$

$$x_T = Dt_S + Fx_S \tag{3.12}$$

$$y_T = y_S \tag{3.13}$$

$$z_T = z_S \tag{3.14}$$

This form was chosen[2] by noting that since we are moving along $x$, the coordinates $y$ and $z$ cannot be affected. We require it to be a linear transformation because non-linear terms (e.g., a $t^2$ term) would make the transformation non-inertial.

We now solve for $A$, $B$, $D$, $F$ by matching important quantities in the two systems and imposing invariance of $c$. Our first two steps are familiar from the Galilean transformation

[2] We also avoid using the letter $C$ to avoid confusion with the speed of light, and the letter $E$ to avoid confusion with energy, which we will be discussing soon.

— we simply require that constant $x$ coordinates in one frame move with speed $v$ in the other frame. Let us focus in particular on the spatial origin:

1. Match the spatial origin of the train frame, $x_T = 0$, with events in the station frame at $x_S = vt_S$:

$$\begin{aligned} x_T &= Dt_S + Fx_S \\ 0 &= Dt_S + Fvt_S \\ \longrightarrow \qquad D &= -Fv \ . \end{aligned} \tag{3.15}$$

   This tells us that our $x$ transformation law can be written $x_T = F(x_S - vt_S)$.

2. Next, match the origin of the station frame ($x_S = 0$) to events in the train frame at $x_T = -vt_T$:

$$\begin{aligned} x_T &= F(x_S - vt_S) \\ -vt_T &= -Fvt_S \end{aligned} \tag{3.16}$$

   This tells us that $t_T = Ft_S$ for events at $x_S = 0$. But we also know

$$t_T = At_S + Bx_S \tag{3.17}$$

   Plugging in $x_S = 0$ and $t_T = Ft_S$, we see that

$$\longrightarrow \qquad F = A \ . \tag{3.18}$$

   We have now pinned down 2 of the 4 unknown coefficients, and the transformation law for $t$ and $x$ reads

$$t_T = At_S + Bx_S \tag{3.19}$$

$$\begin{aligned} x_T &= -Avt_S + Ax_S \\ &= A(x_S - vt_S) \ . \end{aligned} \tag{3.20}$$

   To pin down $A$ and $B$, let us start to use the core physics that is the focus of this lecture: all observers agree that light propagates with speed $c$, so consider the propagation of light as measured in the two reference frames.

3. Imagine a light pulse emitted at $t_S = t_T = 0$, and examine its propagation along the $x_T$ and $x_S$ axes. As seen in the station, it travels with $x_S = ct_S$; as seen on the train, it travels with $x_T = ct_T$:

$$\begin{aligned} x_T &= ct_T \\ A(x_S - vt_S) &= c(At_S + Bx_S) \qquad \text{(Substituting the transformation rules)} \\ A(ct_S - vt_S) &= c(At_S + Bct_S) \qquad \text{(Substituting } x_S = ct_S\text{)} \\ -Avt_S &= Bc^2 t_S \\ \longrightarrow \qquad B &= -\frac{Av}{c^2} \ . \end{aligned} \tag{3.21}$$

   The transformation law now reads

$$t_T = A(t_S - vx_S/c^2) \tag{3.22}$$

$$x_T = A(x_S - vt_S) \ . \tag{3.23}$$

4. Now look at how that pulse travels in the $y$ direction according to observers in the station. They see it moving with $x_S = 0$, $y_S = ct_S$. Observers on the train measure it moving diagonally, following a trajectory in $x_T$ and $y_T$ that satisfies

$$(x_T)^2 + (y_T)^2 = c^2(t_T)^2 \,. \tag{3.24}$$

Substitute $x_T = A(x_S - vt_S)$, $t_T = A(t_S - vx_S/c^2)$, $y_T = y_S = ct_S$, and finally plug in $x_S = 0$:

$$A^2v^2t_S^2 + c^2t_S^2 = c^2A^2t_S^2 \,. \tag{3.25}$$

This is easy to solve for $A$:

$$A = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma \,. \tag{3.26}$$

(If you're being really pedantic you might wonder why we don't consider the negative square root. If you think about the $v = 0$ limit, for which the two coordinate systems should be identical, you see that you need the positive root here.)

Our complete transformation law becomes

$$t_T = \gamma \left(t_S - x_S v/c^2\right) \tag{3.27}$$
$$x_T = \gamma \left(-vt_S + x_S\right) \tag{3.28}$$
$$y_T = y_S \tag{3.29}$$
$$z_T = z_S \,. \tag{3.30}$$

This result is called the *Lorentz transformation.*

A few comments: First, we can make it a bit more symmetric looking by using the definition $\beta = v/c$ we introduced earlier, and by writing $ct_T$ and $ct_S$ as our time variables. This gives our labels for time coordinates the same dimensions (or units) as labels for space. With these minor tweaks, the Lorentz transformation can be written in the matrix form

$$\begin{pmatrix} ct_T \\ x_T \\ y_T \\ z_T \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct_S \\ x_S \\ y_S \\ z_S \end{pmatrix} \,. \tag{3.31}$$

Second, note that Nature doesn't care how we label the axes; we could very well have defined things moving in the $y$ direction or the $z$ direction, or some direction that is at an angle between those directions. If we had the train moving with $\mathbf{v} = v\mathbf{e}_y$, then we would have found

$$\begin{pmatrix} ct_T \\ x_T \\ y_T \\ z_T \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct_S \\ x_S \\ y_S \\ z_S \end{pmatrix} \,. \tag{3.32}$$

You can probably deduce how things look for $\mathbf{v} = v\mathbf{e}_z$.

Finally, how do we invert this transformation? The "brute force" approach is to compute the matrix inverse. However, a little physics helps us see the answer: If the station observer sees the train moving with $\mathbf{v} = v\mathbf{e}_x$, the train observer must see the station moving with

$\mathbf{v} = -v\mathbf{e}_x$. They must develop exactly the same Lorentz transformation, but with the terms linear in $v$ flipped in sign:

$$\begin{pmatrix} ct_S \\ x_S \\ y_S \\ z_S \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct_T \\ x_T \\ y_T \\ z_T \end{pmatrix} . \tag{3.33}$$

It's not hard to show that the matrix in Eq. (3.33) is the inverse of the matrix in Eq. (3.31).

## 3.6 A comment on the road ahead

Much of what we will do in the next few weeks essentially amounts to examining the consequences of the Lorentz transformation, assessing what aspects of physics as we know it hold up and what aspects will need modification. Many of our discussions will involve "thought experiments" of the kind we discuss in the "Consequences" sections above. As such, one can be misled into thinking that much of "Einsteinian" physics is about abstract weird situations like trains that move at nearly the speed of light.

I want to take this moment to make it clear that, though such discussions are useful for understanding important concepts, they are *not* what relativity is about. Like all physics, relativity is a framework *by which we understand the world as we actually measure it*. Special relativity in particular is one of the best-studied theories that we have; its consequences — including the physics of effects like time dilation — have been tested with exquisite accuracy. (Indeed, in a very real sense, *magnetism* is nothing more than a consequence of Coulomb's law of electrostatics plus the Lorentz transformation.) In recent years, the consequences of general relativity have been measured and tested quite thoroughly as well.

We study Einstein's relativity because empirical experience has pointed to the fact that it describes our world exquisitely well. Because you are studying physics, you are likely to encounter people who wish to sell you an alternative[3]. Many of them will claim that the only reason that Einstein gets the attention he is given is because physics has become effectively a priesthood. Some of these folks are bothered by the fact some of the consequences of relativity go against "common sense"; a few claim darker motivations. We will endeavor as much as possible to bring the consequences of relativity into this class, and to keep it grounded in experimental fact. One thing should be clear: if measurements did not agree with Einstein's theories of relativity, we would have discarded them in a heartbeat.

## 3.7 An aside on factors of $c$

The speed of light $c$ pops up so much in this subject that it's very convenient in many analyses to define your units such that $c = 1$. This means that if you measure time in seconds, your basic unit of length is the light second. Amusingly, this means that if you measure time in nanoseconds, your basic unit of length is the light nanosecond, which is almost exactly[4] one foot.

[3] I get at least 5 and as many as 30 emails a week in this theme; I occasionally get hand-written letters and self published books. One guy sent me an adjustable wrench along with his book, I think because he claimed to be "throwing a monkey wrench" into all the "nonsense" that physics departments teach students. It's actually quite a nice wrench. I use it at least twice a year to hook up a hose at my house at the start of summer, and to disconnect it when the weather gets cold.

[4] 1 light nanosecond = 29.9792458 cm = 11.8029 inches = 0.9836 feet.

In my research, I usually set $c = 1$. I raise this in part to warn you; $c = 1$ is so deeply ingrained in my brain now that I tend to leave out factors of $c$ in many important places! Students should be on the lookout for this when I'm writing on the board.