MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
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LECTURE 4
SPACETIME, SIMULTANEITY, AND THE CONSEQUENCES OF LORENTZ

4.1 From space and time to spacetime

The Lorentz transformation shows us that the invariance of c requires space and time to be mixed together; what is "space" for one observer is a mixture of "space" and "time" for another. This should be familiar as far as spatial directions go — what is "left" for one observer can be a mix of "left" and "forward" for another — but mixing time and space like this likely feels somewhat odd. We can no longer think of space and time as purely separate things; we instead describe them as a new, unified entity: *spacetime*. Each inertial observer splits spacetime into space and time; however, *how* they split into space and time differs. This is fundamentally why different inertial observers measure different intervals of time and different interval distances.

One of the tools we will use to examine the geometry of spacetime is the *spacetime diagram*. This is a figure that illustrates how space and time are laid out, as seen by an observer in some particular inertial frame. The convention in making this figures is that time is used for the vertical axis, and space for horizontal axes.

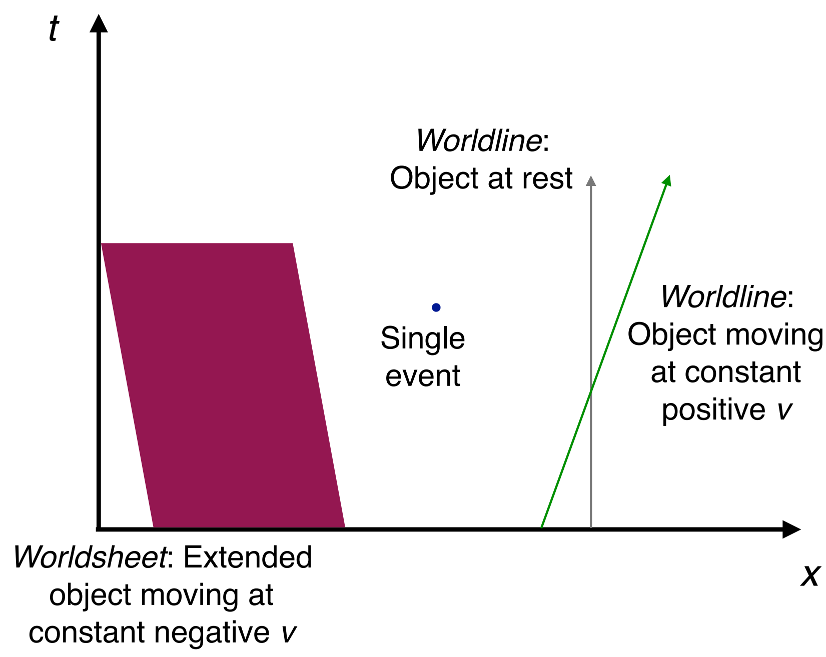
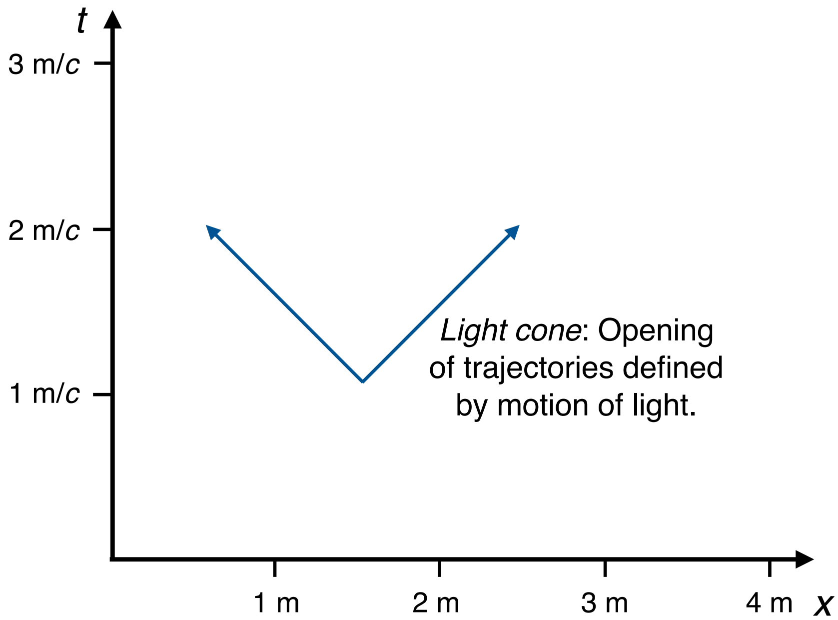


Figure 1: Example of a spacetime diagram. An event is a single point. A *worldline* is the sequence of events swept out by an event as it moves through space and time, with a slope that depends on its velocity in the frame. A *worldsheet* is the set of events swept out by an extended set of events as they move through space and time.

The units of a spacetime diagram's axes are usually chosen so that light moves on 45° lines with respect to the axes of the rest frame:



With such units, a pulse of light, moving through time and projected onto 1 spatial dimension, makes a *lightcone* with an opening angle of 90°. As we will discuss shortly, the lightcone plays an important role in helping us to figure out how events are related to one another.

When making a spacetime diagram, one draws axes corresponding to some particular observer. Suppose we draw the axes of some observer $\mathcal{O}$ who uses coordinates (t, x). How do we represent the coordinates (t', x') of an observer $\mathcal{O}'$ who moves with $\mathbf{v} = v\mathbf{e}_x$ according to $\mathcal{O}$? In other words, what do the (t', x') axes look like as seen by $\mathcal{O}$?

To figure this out, let's look at the transformation rule:

$$ct' = \gamma(ct) - \beta\gamma x \tag{4.1}$$

$$x' = -\beta\gamma(ct) + \gamma x \tag{4.2}$$

The t' axis is defined as the set of events for which $x' = 0$:

$$0 = -\beta\gamma(ct) + \gamma x \qquad \longrightarrow \qquad t = \frac{x}{\beta c} = \frac{x}{v}\ . \tag{4.3}$$

The x' axis is defined by the events for which $t' = 0$:

$$0 = \gamma(ct) - \beta\gamma x \qquad \longrightarrow \qquad t = \frac{\beta x}{c} = \frac{vx}{c^2}\ . \tag{4.4}$$

Figure 2 illustrates the (t', x') axes as seen by $\mathcal{O}$ for an observer moving with $v = \sqrt{3}c/2$.

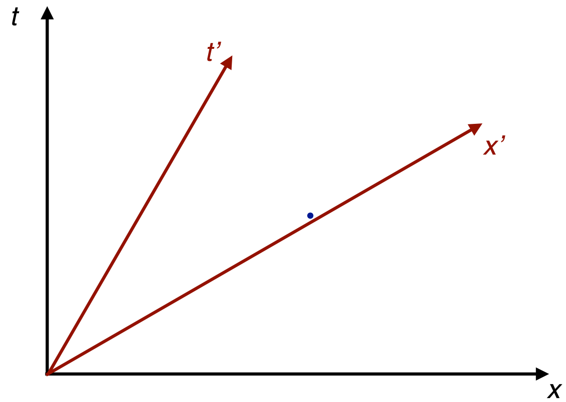


Figure 2: Axes of observer $\mathcal{O}'$ as they appear in the frame of $\mathcal{O}$. The dot represents a particular event.

In this figure, we show a particular event. This event is a geometric object, a single point in spacetime. Although both observers agree on where it is in spacetime, they assign it rather different space and time coordinates. (We will analyze the different labels observers attach to coordinates in some detail shortly.)

We could equally well ask how the axes (t, x) appear according to $\mathcal{O}'$ — we simply use the inverse transformation rule, which yields

$$t' = -\frac{x'}{v} \text{ for the } t \text{ axis}\,, \qquad t' = \frac{-vx'}{c^2} \text{ for the } x \text{ axis.} \tag{4.5}$$

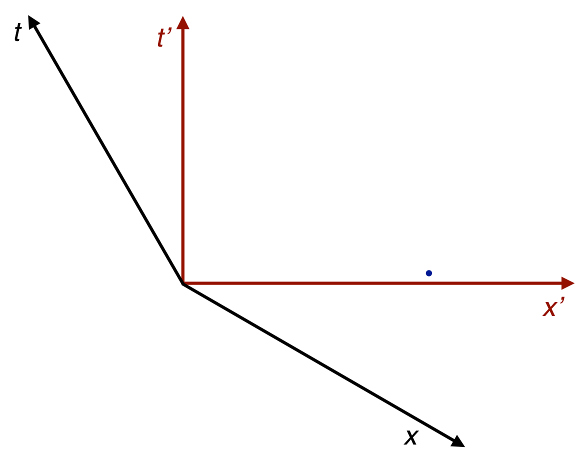


Figure 3: Axes of observer $\mathcal{O}$ as they appear in the frame of $\mathcal{O}'$.

Drawing transformed axes in this way illustrates why length contraction and time dilation arise: Events which are simultaneous — occurring at the same time — in one frame of reference are *not* simultaneous in another frame; events which occur in the same location in one frame do not occur in the same location in another frame. This is the essence of how "space" and "time" are mixed, but "spacetime" remains unified. Different observers agree on "spacetime," but they split it into "space" and "time" in different ways.

4.2 The invariant interval

Imagine two events, labeled A and B. Compute their separation in time and space in some given frame:

$$\Delta t = t_B - t_A \ , \quad \Delta x = x_B - x_A \ , \quad \Delta y = y_B - y_A \ , \quad \Delta z = z_B - z_A \ . \tag{4.6}$$

From these quantities, compute

$$\Delta s^2 \equiv -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \ . \tag{4.7}$$

THEOREM: All inertial observers, in all reference frames, agree on the value of Δs^2.

This theorem is easily proved by simply examining $(\Delta s')^2$, the invariant interval computed using the coordinate separation of the events as measured in some other frame:

$$\Delta t' = t'_B - t'_A \ , \quad \Delta x' = x'_B - x'_A \ , \quad \Delta y' = y'_B - y'_A \ , \quad \Delta z' = z'_B - z'_A \ . \tag{4.8}$$

Let us relate these "primed" separations to the "unprimed" ones using the Lorentz transformation along x we've been using:

$$c\Delta t' = \gamma(c\Delta t) - \gamma\beta\Delta x \ , \tag{4.9}$$

$$\Delta x' = -\gamma\beta(c\Delta t) + \gamma\Delta x \ , \tag{4.10}$$

$$\Delta y' = \Delta y \ , \tag{4.11}$$

$$\Delta z' = \Delta z \ . \tag{4.12}$$

Let us now compute $(\Delta s')^2$:

$$(\Delta s')^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \tag{4.13}$$

$$\begin{aligned} &= -\gamma^2(c\Delta t)^2 + 2\gamma^2\beta(\Delta x)(c\Delta t) - \gamma^2\beta^2(\Delta x)^2 \\ &\quad + \gamma^2\beta^2(c\Delta t)^2 - 2\gamma^2\beta(c\Delta t)(\Delta x) + \gamma^2(\Delta x)^2 \\ &\quad + \Delta y^2 + \Delta z^2 \end{aligned} \tag{4.14}$$

$$= -c^2\Delta t^2\left[\gamma^2(1-\beta^2)\right] + \Delta x^2\left[\gamma^2(1-\beta^2)\right] + \Delta y^2 + \Delta z^2 \tag{4.15}$$

$$= -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{4.16}$$

$$= \Delta s^2 \ . \tag{4.17}$$

The first line of this is just the definition of $(\Delta s')^2$. To go to the second line, we've used the Lorentz transformation to express the primed-frame quantities in terms of unprimed-frame quantities. To go to the third line, we gather terms together, canceling out the terms that involve $(\Delta x)(c\Delta t)$, and gathering common factors of Δx^2 and $c^2\Delta t^2$. To go to the fourth line, we used the fact that $\gamma = 1/\sqrt{1-\beta^2}$. That line reproduces Δs^2, demonstrating[1] that this quantity is a Lorentz invariant.

We are going to do a lot with Δs^2, a quantity that we call the *invariant interval* (often abbreviated to just the "interval"). To start, it's worth noting that perhaps the most important property of this quantity is whether it is negative, positive, or zero:

[1] It is easy to verify that this works for the transformation along any axis. In another lecture or two, we will introduce notation that makes proving the invariance of quantities like this really easy for *any* Lorentz transformation.

- $\Delta s^2 < 0$: in this case, the interval is dominated by Δt. We say that the two events have **timelike** separation. When $\Delta s^2 < 0$, it means that we can find some Lorentz frame in which the events A and B have the same spatial position (i.e., in that frame $x_A = x_B$, $y_A = y_B$, $z_A = z_B$); the events are only separated by time in that frame. We define $\Delta\tau \equiv \sqrt{-\Delta s^2}/c$ to be the time elapsed between events A and B in that frame. We call $\Delta\tau$ the **proper-time interval** — it is the interval of time measured by the observer who is at rest in the frame in which A and B are co-located.

  It's worth noting that if the interval between two events is timelike, then one can imagine a signal which travels with speed $v < c$ that connects them.

- $\Delta s^2 > 0$: the interval here is dominated by $\Delta x^2 + \Delta y^2 + \Delta z^2$, and we say that the two events have **spacelike** separation. In this case, we can find a Lorentz frame in which events A and B are simultaneous; Δs is the distance between these events in that frame. We call Δs the **proper separation** of A and B.

- $\Delta s^2 = 0$: in this case, we find that $c\Delta t = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ — events A and B have a lightlike or "null" separation. If $\Delta s^2 = 0$, then these events can be connected by a light pulse.

The last point helps us to see that the value of Δs^2 is very closely connected to the properties of the lightcone mentioned earlier. Suppose a flash of light is emitted from event A. If the interval between A and another event is negative, $\Delta s^2 < 0$, then the other event must be *inside* the lightcone. If the interval is positive, then the event must be *outside* the lightcone. And if $\Delta s^2 = 0$, then the other event must be on the light cone itself. Figure 4 illustrates how these notions connect to the lightcone.

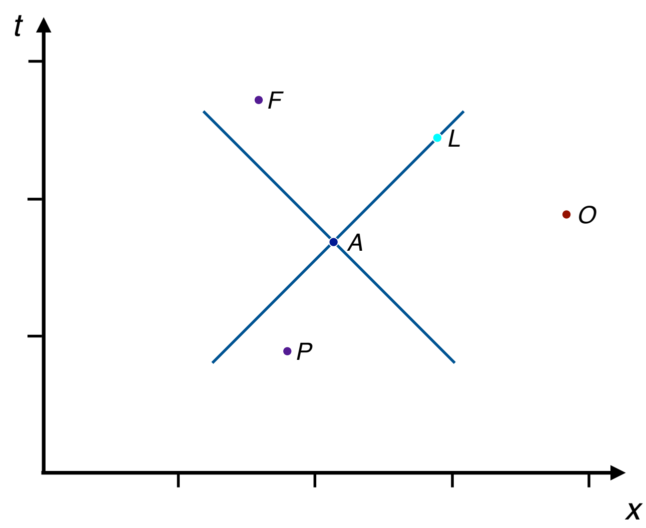


Figure 4: The intervals between events A and F and events A and P are timelike: $\Delta s^2_{AF} < 0$, $\Delta s^2_{AP} < 0$. In all frames, event F has time coordinate greater than the time coordinate of event A: $t_F > t_A$. Event F is unambiguously in the *future* of event A. Likewise, event P has time coordinate less than the time coordinate of event A: $t_P < t_A$ in all frames. Event P is unambiguously in the *past* of event A. Events A and O have a spacelike interval: $\Delta s^2_{AO} > 0$. Event O is neither in the future nor the past of A; it is "elsewhere," so the time-ordering of these events is not invariant. Events A and L have a lightlike or null interval: $\Delta s^2_{AL} = 0$. These events are connected by a light beam in all reference frames.

4.3 The geometry of spacetime

The relationship $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ essentially expresses the Pythagorean theorem for spacetime. For intuition, consider the Pythagorean theorem purely in space. On a flat two-dimensional surface, a right triangle whose sides are Δx and Δy has a hypotenuse whose length is determined from $\Delta s^2 = \Delta x^2 + \Delta y^2$. In three dimensions, the distance from (x, y, z) to $(x + \Delta x, y + \Delta y, z + \Delta z)$ is given by $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$.

In spacetime, it turns out to be extremely useful to regard $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ as expressing an invariant notion of "distance squared" between two events. Students usually want to know "Why does the $c^2\Delta t^2$ have a minus sign?" The best answer I can give is that this is how the geometry of the universe works. The fact that time enters Δs^2 with a different sign from space reflects the fact that time is fundamentally quite different from the other directions of spacetime. We can forward and backward; we can move left and right; we can move up and down. But we can only move toward the future — we cannot step back to the past.

Indeed, the whole notion of "past" and "future" depends on events' separation in spacetime. If two events are timelike or lightlike separated, then one can describe one event as being the future, and one in the past. Although the specific time coordinates assigned to these events will vary by reference frame, the time *ordering* of these events is invariant: if $t_F > t_A$ in one frame, and if the interval between events A and F is timelike or lightlike, then $t_F > t_A$ in *all* reference frames. However, if two events are spacelike separated, then their time ordering depends on reference frame. Consider the situation shown in Figure 5:

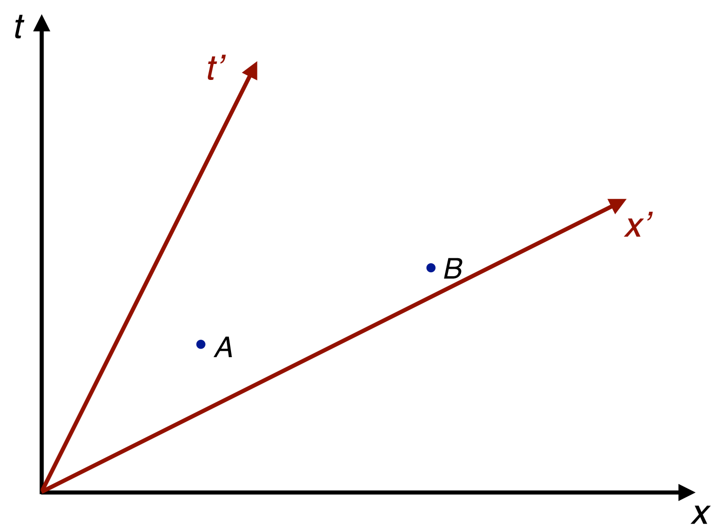


Figure 5: Observer $\mathcal{O}$ measures coordinates for events A and B using the (t, x) axes. Observer $\mathcal{O}'$, who travels with velocity $\mathbf{v} = (c/2)\mathbf{e}_x$ according to $\mathcal{O}$, measures coordinates for these events using the (t', x') axes.

Suppose observer $\mathcal{O}$ measures these events at the coordinates $(t_A, x_A) = (2\,\mathrm{sec}, 2\,\mathrm{lightsec})$, $(t_B, x_A) = (3\,\mathrm{sec}, 5\,\mathrm{lightsec})$. So, for observer $\mathcal{O}$, event A happens first. However, the invariant interval between these events,

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 = -(1\,\mathrm{lightsec})^2 + (3\,\mathrm{lightsec})^2 = 8\,\mathrm{lightsec}^2 \ , \tag{4.18}$$

is *positive* — these events are spacelike separated, so different observers may very well order them differently.

Let's use the Lorentz transformation to compute the events' coordinates according to $\mathcal{O}'$. Given the relative speed $c/2$, we have $\gamma = 2/\sqrt{3}$, $\beta = 1/2$. Applying the Lorentz transformation, we find

$$ct'_A = \gamma t_A - \beta\gamma x_A = \left(4/\sqrt{3} - 2/\sqrt{3}\right) \text{ lightsec} = \frac{2}{\sqrt{3}} \text{ lightsec} , \tag{4.19}$$

$$x'_A = -\beta\gamma t_A + \gamma x_A = \left(-2/\sqrt{3} + 2/\sqrt{3}\right) \text{ lightsec} = \frac{2}{\sqrt{3}} \text{ lightsec} ; \tag{4.20}$$

$$ct'_B = \gamma t_B - \beta\gamma x_B = \left(6/\sqrt{3} - 5/\sqrt{3}\right) \text{ lightsec} = \frac{1}{\sqrt{3}} \text{ lightsec} , \tag{4.21}$$

$$x'_B = -\beta\gamma t_B + \gamma x_B = \left(-3/\sqrt{3} + 10/\sqrt{3}\right) \text{ lightsec} = \frac{7}{\sqrt{3}} \text{ lightsec} . \tag{4.22}$$

$$\begin{aligned} \longrightarrow \quad (t'_A, x'_A) &= \left(\frac{2}{\sqrt{3}} \text{ sec}, \frac{2}{\sqrt{3}} \text{ lightsec}\right) \\ &\simeq (1.15\,\text{sec}, 1.15\,\text{lightsec}) \end{aligned} \tag{4.23}$$

$$\begin{aligned} \longrightarrow \quad (t'_B, x'_B) &= \left(\frac{1}{\sqrt{3}} \text{ sec}, \frac{7}{\sqrt{3}} \text{ lightsec}\right) \\ &\simeq (0.577\,\text{sec}, 4.04\,\text{lightsec}) . \end{aligned} \tag{4.24}$$

Notice that $t'_A > t'_B$: the order of the events is reversed according to observer $\mathcal{O}'$. Using these numbers, it is not difficult to show that $\mathcal{O}'$ nonetheless finds $\Delta s^2 = 8\,\text{lightsec}^2$.