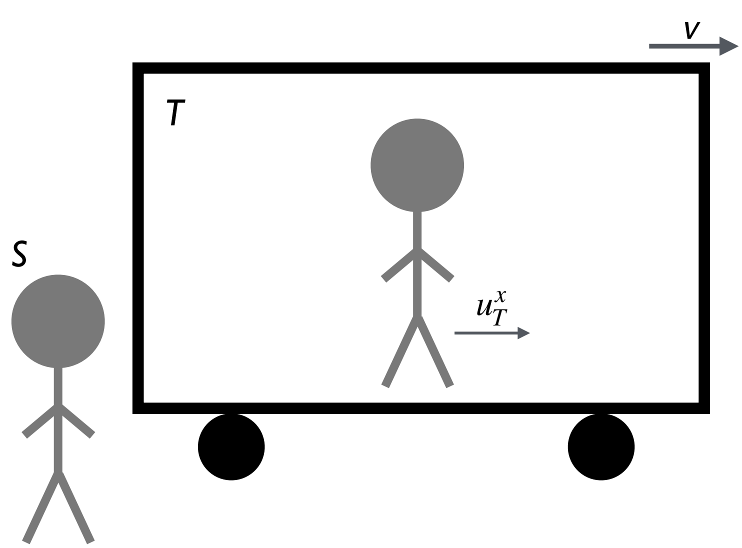
Massachusetts Institute of Technology
Department of Physics
8.033 Fall 2021

Lecture 6
Kinematics in spacetime

6.1 Transforming velocities

With what we've done so far, we've started to develop a good understanding of length, time, and geometry in spacetime. This is a good start for us to begin understanding physics in special relativity, but it's just a start.

In this lecture, we start examining *kinematics* — the properties of moving bodies, and how these properties transform between different reference frames. Let's begin by looking at velocity. Consider frame T, tied to a train, and consider someone walking inside that train. This train is moving with velocity $\mathbf{v} = v\mathbf{e}_x$ as seen by an observer who is at rest in the station frame S. The person is seen to walk with speed u^x_T, also in the x direction, by an observer at rest in frame T. (Comment: we will try as much as possible to use the letter u to stand for speeds inside a particular frame; we will try to use v to describe the speeds and velocities between two different frames.)



What is the speed u^x_S that observers in frame S measure? In Newtonian physics, we would just add the velocities in frame T to the velocity that frame T has relative to S. To get u^x_S in a world in which all observers agree that light moves at speed c, we work this out using the Lorentz transformation. On the train, we know that in a time interval Δt_T observer T moves through a distance $\Delta x_T = u^x_T \Delta t_T$. Both the time interval and the space interval are affected by the transformation:

$$\begin{aligned} u^x_S = \frac{\Delta x_S}{\Delta t_S} &= \frac{\gamma(\Delta x_T + v\Delta t_T)}{\gamma(\Delta t_T + v\Delta x_T/c^2)} \\ &= \frac{(\Delta x_T/\Delta t_T + v)}{(1 + \frac{v\Delta x_T}{c^2\Delta t_T})} \\ &= \frac{u^x_T + v}{1 + u^x_T v/c^2} \,. \end{aligned} \tag{6.1}$$

This formula has an interesting consequence: using it, we can prove that we can never add sub-light speeds to get a speed that exceeds the speed of light. You will work this out in detail on a problem set, but to see the general idea, imagine that $u^x_T = v = 0.9c$. Then,

$$u^x_S = \frac{0.9c + 0.9c}{1 + (0.9c)(0.9c)/c^2} = \frac{1.8c}{1.81} = 0.9945c \,. \tag{6.2}$$

How do components of the velocity perpendicular to the frames' relative motion transform? Imagine that the person on the train has motion along the y direction as well, so that in Δt_T they move through $\Delta y_T = u^y_T \Delta t_T$. Then,

$$\begin{aligned} u^y_S = \frac{\Delta y_S}{\Delta t_S} &= \frac{\Delta y_T}{\gamma(\Delta t_T + v\Delta x_T/c^2)} \\ &= \frac{u^y_T}{\gamma(1 + u^x_T v/c^2)} \,. \end{aligned} \tag{6.3}$$

(Note that the factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ — it only depends on the relative speed v of the two frames, it does not involve the velocity $\mathbf{u}$.) If the person on the train has velocity along the z direction, then it transforms like Eq. (6.3) as well, replacing u^y with u^z.

6.2 Momentum I: Uh oh ... did we break physics?

A lesson of the previous section is that how velocities add is "weird" as compared to Newtonian expectations. These expectations follow the logic of Galilean relativity, so it should not too surprising that things change when we impose the rule that c is the same to all observers. However, our laws of classical mechanics have implicitly assumed Galilean relativity. What happens to important principles like conservation of momentum when we "update" our laws for how velocities add?

Let us first review how conservation of momentum works in Newtonian physics. Suppose that we have N_i bodies that come together in some fashion, interact, and then have N_f bodies in the final state. Conservation of momentum tells us that

$$\sum_{j=1}^{N_i} m_j^{\rm initial} \mathbf{u}_j^{\rm initial} = \sum_{j=1}^{N_f} m_j^{\rm final} \mathbf{u}_j^{\rm final} \,. \tag{6.4}$$

As long as

$$\sum_{j=1}^{N_i} m_j^{\rm initial} = \sum_{j=1}^{N_f} m_j^{\rm final} \,, \tag{6.5}$$

this relation holds in all Galilean reference frames.

Let's take a look at what happens when we examine this law in Lorentzian reference frames. Let's consider something really simple: two particles, A and B, of identical mass that collide and rebound elastically. First, examine this situation in the *center of momentum* frame, i.e. the frame in which the net momentum of the system is zero:

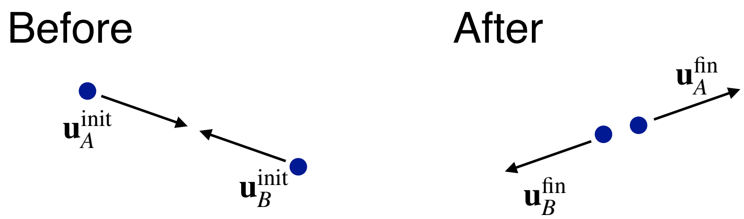


Figure 1: Elastic collision of identical bodies in the center of momentum frame.

The bodies' velocities are given by $\mathbf{u}_A^{\rm init} = u^x\mathbf{e}_x - u^y\mathbf{e}_y$, $\mathbf{u}_B^{\rm init} = -u^x\mathbf{e}_x + u^y\mathbf{e}_y$ before the collision. Afterwards, we have $\mathbf{u}_A^{\rm fin} = u^x\mathbf{e}_x + u^y\mathbf{e}_y$, $\mathbf{u}_B^{\rm init} = -u^x\mathbf{e}_x - u^y\mathbf{e}_y$. Because $m_A = m_B$, we can see that momentum is clearly conserved: It is zero both before and after the collision.

Let's examine this from another frame of reference. Suppose we examine this collision from a frame that moves with velocity $\mathbf{v} = -u^x\mathbf{e}_x$ with respect to the center of momentum frame. In this frame, the horizontal motion of particle B is canceled out. What are the velocity vectors in this frame? We can find out by using the velocity addition formulas we just worked out. Let's do the x components first:

$$u_A^{x'} = \frac{u^x + u^x}{1 + (u^x)^2/c^2} = \frac{2u^x}{1 + (u^x)^2/c^2} , \tag{6.6}$$

$$u_B^{x'} = \frac{u^x - u^x}{1 - (u^x)^2/c^2} = 0 . \tag{6.7}$$

Notice that the horizontal component of momentum is no longer zero. That is not surprising: we've moved into a frame in which the entire system is moving in the $+x$ direction, so we expect the system to have momentum along x.

Next, look at the y components:

$$u_A^{y'} = -\frac{u^y}{\gamma(1 + (u^x)^2/c^2)} = -\frac{u^y\sqrt{1 - (u^x)^2/c^2}}{1 + (u^x)^2/c^2} , \tag{6.8}$$

$$u_B^{y'} = \frac{u^y}{\gamma(1 - (u^x)^2/c^2)} = \frac{u^y}{\sqrt{1 - (u^x)^2/c^2}} . \tag{6.9}$$

(We've used $\gamma = 1/\sqrt{1 - (u^x)^2/c^2}$ here.) Notice that the velocity components in the vertical direction are no longer equal and opposite. If the vertical component of a particle's momentum is given by computing $mu^{y'}$, we have a problem: *The system appears to have acquired momentum in the y direction by moving into a new frame that is moving in the $-x$ direction with respect to the center of momentum frame.*

Our hypothesis that c is the same to all observers, which led to our new velocity addition rules, appears to have broken momentum.

6.3 Momentum II: No, physics isn't broken, but Newtonian expectations are incomplete

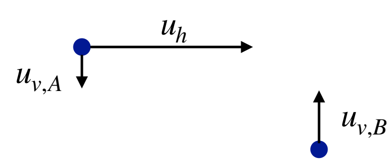
This appears disturbing. However, as we have seen, our hypothesis of the invariance of c is *approximately* consistent with Galilean coordinate transformations; perhaps the root cause of our issue is that Newtonian momentum (which respects Galilean relativity) is itself an approximation to a more fundamental quantity.

Let us postulate that momentum is defined by

$$\mathbf{p} = \alpha(u) m\mathbf{u} \ . \tag{6.10}$$

The function $\alpha(u)$ is some kind of scalar which corrects the magnitude of momentum, and only depends on the magnitude of the body's velocity $\mathbf{u}$. Let us re-examine the collision from the Lorentz frame in which particle B has no horizontal motion:

Before collision, frame in which *B* only moves vertically



To simplify some of the analysis which will follow later, we've introduced new labels for the velocity components of these bodies. Referring to Eqs. (6.6), (6.8), and (6.9) using the original center-of-momentum frame velocity components, we have

$$u_h = \frac{2u^x}{1+(u^x)^2/c^2} \ , \quad u_{v,A} = -\frac{u^y\sqrt{1-(u^x)^2/c^2}}{1+(u^x)^2/c^2} \ , \quad u_{v,B} = \frac{u^y}{\sqrt{1-(u^x)^2/c^2}} \ . \tag{6.11}$$

These velocity components turn out to be nicely related to one another. Notice that

$$u_{v,A} = -u_{v,B}\left(\frac{1-(u^x)^2/c^2}{1+(u^x)^2/c^2}\right) \ . \tag{6.12}$$

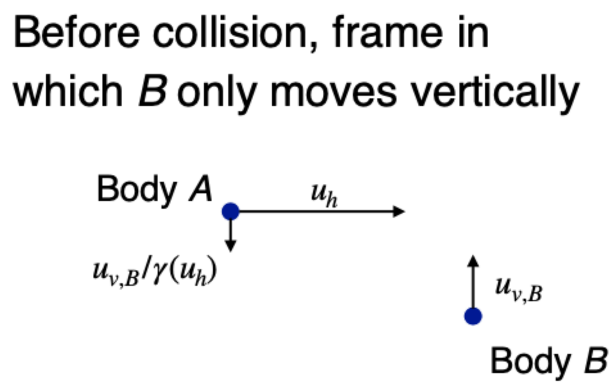
The factor in parentheses turns out be related to u_h in an interesting way:

$$\begin{aligned}\gamma(u_h) &= \frac{1}{\sqrt{1-(u_h)^2/c^2}} = \left(1 - \frac{4(u^x)^2/c^2}{(1+(u^x)^2/c^2)^2}\right)^{-1/2} \\ &= \left(\frac{1-2(u^x)^2/c^2+(u^x)^4/c^4}{1+2(u^x)^2/c^2+(u^x)^4/c^4}\right)^{-1/2} \\ &= \frac{1+(u^x)^2/c^2}{1-(u^x)^2/c^2} \ . \end{aligned} \tag{6.13}$$

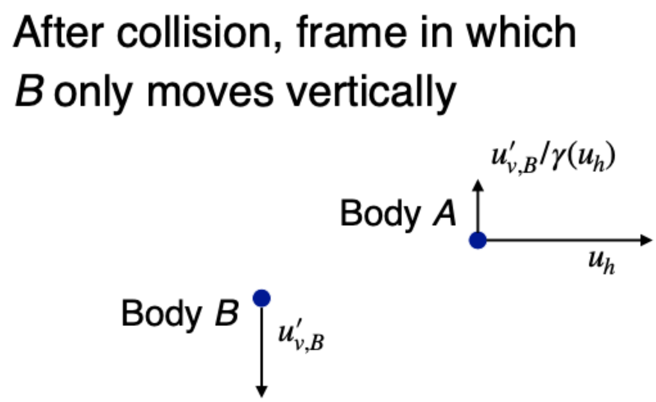
This tells us that

$$u_{v,A} = -u_{v,B}/\gamma(u_h) \ . \tag{6.14}$$

Let's take advantage of this to redo the figure of the collision in this frame using only the velocity components u_h and $u_{v,B}$:



If momentum is conserved, then we expect the situation after the collision to look as follows:



The logic by which we have sketched this is that the horizontal components of the bodies' motion cannot be affected by the collision, so body A continues moving to the right with speed u_h, and body B continues to have no horizontal motion. The vertical motions reverse in direction, and we leave open the possibility that the magnitudes associated with the vertical speed might be affected.

We now demand conservation of our postulated modification to momentum: both components of $\mathbf{p} = \alpha(u)m\mathbf{u}$ must be the same before and after the collision. Let us first look at the horizontal component, for which the only contribution comes from body A:

$$\alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right) m u_h = \alpha\left(\sqrt{(u_h)^2 + (u'_{v,B}/\gamma(u_h))^2}\right) m u_h \; . \tag{6.15}$$

The only way that this equation can hold independent of the function $\alpha(u)$ (whose nature we don't yet know) is if $u'_{v,B} = u_{v,B}$. In other words, as measured in this frame, the magnitude of the vertical components' of the bodies' velocities remains the same, those components simply change direction.

Let's now require that the vertical components be conserved:

$$\alpha(u_{v,B})mu_{v,B} - \alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right)mu_{v,B}/\gamma(u_h) =$$
$$- \alpha(u_{v,B})mu_{v,B} + \alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right)mu_{v,B}/\gamma(u_h)$$
$$\longrightarrow \qquad \alpha\left(\sqrt{(u_h)^2 + (u_{v,B}/\gamma(u_h))^2}\right) = \gamma(u_h)\alpha(u_{v,B}) \; . \tag{6.16}$$

We may safely assume that $\alpha(0) = 1$ — this is a way of insuring that the formula recovers the Newtonian limit, which is at the very least a very good approximation. With this in mind, examine Eq. (6.16) with $u_{v,B} \to 0$:

$$\alpha(u_h) = \gamma(u_h) \; . \tag{6.17}$$

In other words, the factor $\alpha(u)$ that we postulated was needed indeed "restores" a notion of momentum that is conserved provided it is the special relativistic γ factor.

To conclude and summarize, the momentum that is conserved when the universe respects Lorentz covariance is given by

$$\mathbf{p} = \gamma(u)m\mathbf{u} \; . \tag{6.18}$$

6.4 Kinetic energy

In Newtonian physics, the change in kinetic energy is the work done on a body: Integrating from some initial position $\mathbf{x}_i$ to a final position $\mathbf{x}_f$, we have

$$\begin{aligned} K_f - K_i &= \int_i^f \frac{d\mathbf{p}}{dt} \cdot d\mathbf{x} = \int_i^f \frac{d}{dt}\left(m\mathbf{u}\right) \cdot \mathbf{u}\, dt = m \int_i^f \mathbf{u} \cdot d\mathbf{u} \\ &= \frac{1}{2}m\left(u_f^2 - u_i^2\right) \; . \end{aligned} \tag{6.19}$$

Let's define relativistic kinetic energy in exactly the same way, but replace the Newtonian formula for momentum with the version we just derived:

$$\begin{aligned} K_f - K_i &= \int_i^f \frac{d\mathbf{p}}{dt} \cdot d\mathbf{x} = \int_i^f \frac{d}{dt}\left[\gamma(u)m\mathbf{u}\right] \cdot \mathbf{u}\, dt \\ &= m \int_i^f \mathbf{u} \cdot d\left[\frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right] \; . \end{aligned} \tag{6.20}$$

The final integrand that we have derived can be manipulated further:

$$\mathbf{u} \cdot d\left[\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}\right] = d\left[\frac{u^2}{\sqrt{1-u^2/c^2}}\right] - \frac{\mathbf{u} \cdot d\mathbf{u}}{\sqrt{1-u^2/c^2}} \; . \tag{6.21}$$

Using this, we find

$$\begin{aligned} K_f - K_i &= \left.\frac{mu^2}{\sqrt{1-u^2/c^2}}\right|_i^f - m\int_i^f \frac{\mathbf{u} \cdot d\mathbf{u}}{\sqrt{1-u^2/c^2}} \\ &= \left.\frac{mu^2}{\sqrt{1-u^2/c^2}}\right|_i^f + \left.mc^2\sqrt{1-u^2/c^2}\right|_i^f \; . \end{aligned} \tag{6.22}$$

To write our final answer, let's assume that the initial velocity $\mathbf{u}_i = 0$, and define $\mathbf{u}_f \equiv \mathbf{u}$. Since the initial velocity is zero, the initial kinetic energy $K_i = 0$. We then set $K_f \equiv K$ and finally obtain for the kinetic energy of the system

$$K = \frac{mu^2}{\sqrt{1 - u^2/c^2}} + mc^2\sqrt{1 - u^2/c^2} - mc^2 \; , \tag{6.23}$$

or

$$\begin{aligned} K &= \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2 \\ &= [\gamma(u) - 1]\, mc^2 \; . \end{aligned} \tag{6.24}$$

We interpret this quantity by defining the body to have a *total* energy $E = \gamma mc^2$, and then write

$$E = K + mc^2 \; , \tag{6.25}$$

where mc^2 is interpreted as the body's **rest energy** — that is, energy which the body possesses even when it is not in motion.

It's fair to say that Eq. (6.25) with $K = 0$ is the most famous physics equation in the world. Now you can see that it arises as a consequence of the fact that the hypothesis of c being the same to all observers forced us to replace the Galilean transformation with the Lorentz transformation. This in turn mandated an adjustment to the definition of momentum, from which this famous result emerged.

6.5 Aside: "Relativistic mass" and why we generally don't use it anymore

In some older texts, you will see the energy and momentum defined as follows:

$$E = m(u)c^2 \; , \qquad \mathbf{p} = m(u)\mathbf{u} \; , \tag{6.26}$$

where they have defined $m(u) = \gamma(u)m$, the "relativistic mass" of the body whose rest mass is m. This definition rarely appears in modern relativity texts. Instead, the only "mass" used to define a body is its rest mass. The main reason for this is that m is an *invariant* — different observers assign a different energy to the body, depending on its speed u in their rest frame, but they all agree that the body's mass is m (and its energy is mc^2) in its own rest frame. As we will see in the next lecture, this invariant plays a particularly important rule in helping us to define a 4-vector which will prove to be extremely useful in helping us to keep track of energy and momentum in relativistic physics.