Massachusetts Institute of Technology
Department of Physics
8.033 Fall 2021

Lecture 12
A covariant formulation of electromagnetics (part II)

12.1 The field equations

In the previous lecture, we showed that the Lorentz force law written using 3-vectors,

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \;, \tag{12.1}$$

is exactly equivalent to the 4-force law

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\beta}u_\beta \;, \tag{12.2}$$

provided that the *Faraday tensor* components are related to the electric and magnetic field components according to

$$F^{\alpha\beta} \doteq \begin{pmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & B^z & -B^y \\ -E^y/c & -B^z & 0 & B^x \\ -E^z/c & B^y & -B^x & 0 \end{pmatrix} \;. \tag{12.3}$$

More specifically, we found that the spatial components of $dp^\alpha/d\tau$ correspond exactly to the 3-force $\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$, and that the 0 or timelike component tells us about the work that is done on a charge by the electric field.

In this lecture, we are going to turn to a study of the field equations: how do we make the set of Maxwell equations,

$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0 \;, \qquad \nabla\cdot\mathbf{B} = 0 \;, \tag{12.4}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \;, \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \;, \tag{12.5}$$

fit into this framework?

The first thing we want to do is massage these equations a little bit. Notice that half of the Maxwell equations involve sources, either ρ or $\mathbf{J}$; the other half only involve the fields themselves. Let's reorganize the equations to emphasize this structure:

$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0 \;, \qquad \nabla\times\mathbf{B} - \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J} \;; \tag{12.6}$$

$$\nabla\cdot\mathbf{B} = 0 \;, \qquad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \;. \tag{12.7}$$

We have put all terms that involve the fields onto the left-hand side of these equations, and set them so that the right-hand side is either "source" (ρ or $\mathbf{J}$) or zero. Notice that there are four sourced equations (one divergence of $\mathbf{E}$, three components of the curl of $\mathbf{B}$), and four source-free equations (one divergence of $\mathbf{B}$, three components of the curl of $\mathbf{E}$).

12.1.1 Half of the Maxwell equations

Let's start by just taking derivatives of the Faraday tensor. By contracting a derivative on one of the indices, we'll generate four different terms, one for each value of the remaining free index:

$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = \partial_\beta F^{\alpha\beta} \ . \tag{12.8}$$

(Why contract on the second index? Strictly speaking, it doesn't matter much — because $F^{\alpha\beta}$ is antisymmetric, we'd just get a minus sign if we contracted on the first one.)

Let's go into a Lorentz frame and see what $\partial_\beta F^{\alpha\beta}$ looks like as α goes over its free range:

$$\begin{aligned} \alpha = 0: \qquad \partial_\beta F^{0\beta} &= \frac{\partial}{\partial x}\left(\frac{E^x}{c}\right) + \frac{\partial}{\partial y}\left(\frac{E^y}{c}\right) + \frac{\partial}{\partial z}\left(\frac{E^z}{c}\right) \\ &= \frac{1}{c}\nabla\cdot\mathbf{E} \ . \end{aligned} \tag{12.9}$$

In other words, up to a factor of $1/c$, the $\alpha = 0$ component of $\partial_\beta F^{\alpha\beta}$ looks just like the divergence of $\mathbf{E}$, and so produces the left-hand side of one of the sourced Maxwell equations.

Let's look at the other values of α:

$$\begin{aligned} \alpha = 1: \qquad \partial_\beta F^{1\beta} &= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{E^x}{c}\right) + \frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z} \\ &= -\frac{1}{c^2}\frac{\partial E^x}{\partial t} + (\nabla\times\mathbf{B})^x \\ &= -\mu_0\epsilon_0\frac{\partial E^x}{\partial t} + (\nabla\times\mathbf{B})^x \ . \end{aligned} \tag{12.10}$$

(We've used the fact that $c = 1/\sqrt{\mu_0\epsilon_0}$ here.) This analysis shows that the $\alpha = 1$ component of $\partial_\beta F^{\alpha\beta}$ produces the left-hand side of another one of the source Maxwell equations. It's not too hard to show that the $\alpha = 2$ and $\alpha = 3$ components produce the remaining two left-hand sides:

$$\alpha = 2: \qquad \partial_\beta F^{2\beta} = -\mu_0\epsilon_0\frac{\partial E^y}{\partial t} + (\nabla\times\mathbf{B})^y \ , \tag{12.11}$$

$$\alpha = 3: \qquad \partial_\beta F^{3\beta} = -\mu_0\epsilon_0\frac{\partial E^z}{\partial t} + (\nabla\times\mathbf{B})^z \ . \tag{12.12}$$

To get the right-hand side of the sourced Maxwell equations, recall a few lectures ago that we defined the 4-vector $\vec{J}$ whose time-like component $J^t = c\rho$, but whose space-like components are the "normal" 3-vector current density. Comparison of Eq. (12.6) with Eqs. (12.10) – (12.12) suggest that the form we want is

$$\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha \ . \tag{12.13}$$

It's pretty clear that this form works perfectly for $\alpha = 1, 2, 3$. Does it also work for $\alpha = 0$? Let's check: using Eq. (12.9),

$$\partial_\beta F^{0\beta} = \mu_0 J^0 \qquad \text{becomes} \qquad \frac{1}{c}\nabla\cdot\mathbf{E} = \mu_0 c\rho \ . \tag{12.14}$$

Multiplying both sides by c and using $c = 1/\sqrt{\mu_0\epsilon_0}$, this becomes

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \ . \tag{12.15}$$

So it works! We've found that *half* of the Maxwell equations — the half that have source terms, either charge density ρ or current density $\mathbf{J}$ — are equivalent to the equation

$$\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha \ . \tag{12.16}$$

12.1.2 The other half of the Maxwell equations

What about the other half of the Maxwell equations — how do we get the ones that don't have a source? There's no way that we can get those equations just by taking derivatives of $F^{\alpha\beta}$. Differentiating this quantity can only duplicate the derivatives that we've already worked out and used to get the sourced Maxwell equations. We need a different way of organizing the fields.

The way we get there is by thinking about how to organize the electric and magnetic fields into an antisymmetric tensor. Suppose we take $F^{\alpha\beta}$ swap the electric and magnetic field components as follows:

$$F^{\alpha\beta}\left(\mathbf{E}/c \to \mathbf{B}\ ,\ \mathbf{B} \to -\mathbf{E}/c\right) = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z/c & E^y/c \\ -B^y & E^z/c & 0 & -E^x/c \\ -B^z & -E^y/c & E^x/c & 0 \end{pmatrix} \equiv G^{\alpha\beta} \ . \tag{12.17}$$

This quantity is known as the *dual*[1] Faraday tensor. It has the same symmetries as the Faraday tensor; and, if you apply the rule $\mathbf{E}/c \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}/c$ to the rules for Lorentz transforming the fields, you find that they are unchanged. [You can test this by applying the rule to Eqs. (11.37) and (11.38) from Lecture 11]. The dual Faraday tensor does not[2], however, give us a force law.

If we differentiate $G^{\alpha\beta}$, we get field derivatives that differ from those that come from differentiating $F^{\alpha\beta}$. Let's go through a few examples of $\partial_\beta G^{\alpha\beta}$:

$$\begin{aligned} \alpha = 0: \qquad \partial_\beta G^{0\beta} &= \frac{\partial B^x}{\partial x} + \frac{\partial B^y}{\partial y} + \frac{\partial B^z}{\partial z} \\ &= \nabla \cdot \mathbf{B}\ ; \end{aligned} \tag{12.18}$$

$$\begin{aligned} \alpha = 1: \qquad \partial_\beta G^{1\beta} &= -\frac{1}{c}\frac{\partial B^x}{\partial t} - \frac{1}{c}\frac{\partial E^z}{\partial y} + \frac{1}{c}\frac{\partial E^y}{\partial z} \\ &= -\frac{1}{c}\left[\frac{\partial B^x}{\partial t} + (\nabla \times \mathbf{E})^x\right] \ . \end{aligned} \tag{12.19}$$

The $\alpha = 2$ and $\alpha = 3$ components duplicate the y and z components of the curl $\mathbf{E}$ part of Eq. (12.7). Putting this all together, we see that

$$\partial_\beta G^{\alpha\beta} = 0 \tag{12.20}$$

[1]You might find the way that we derived this dual tensor to be somewhat schematic. There is in fact a more rigorous way of doing this which takes advantage of a 4-index version of the Levi-Civita symbol you used on problem set 3: by appropriately combining $F^{\alpha\beta}$ with $\epsilon_{\alpha\beta\gamma\delta}$ (an object which generalizes ϵ_{ijk} to spacetime) and the metric $\eta_{\alpha\beta}$, we can build the tensor $G^{\alpha\beta}$. For the purpose of 8.033, the schematic approach is good enough.

[2]Interestingly, this tensor *would* be involved in a force law if there were magnetic charges as well as electric charges. Perhaps something to explore on a problem set...

is exactly what we need to write the source-free Maxwell equations in a covariant way.

To summarize: our original presentation of the Maxwell equations, Eqs. (12.4) and (12.5) are not wrong, but are formulated in such a way that they use information specific to some particular Lorentz frame. The fields $\mathbf{E}$ and $\mathbf{B}$ are particular to that observer, as is the charge density ρ and current density $\mathbf{J}$, as is the notion of space and time they use to take their derivatives. These equations are exactly equivalent to the covariant formulation

$$\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha \ , \qquad \partial_\beta G^{\alpha\beta} = 0 \ . \tag{12.21}$$

For our present purpose, Eq. (12.21) is preferred to Eqs. (12.4) and (12.5) because it shows us how to write these equations in a way that is formulated for a different Lorentz observer. If the coordinates $x^{\alpha'}$ are used by $\mathcal{O}'$, then we know that their formulation of Maxwell's equations looks like

$$\partial_{\beta'} F^{\alpha'\beta'} = \mu_0 J^{\alpha'} \ , \qquad \partial_{\beta'} G^{\alpha'\beta'} = 0 \ . \tag{12.22}$$

We can get all the "prime frame" quantities by just appropriate correcting things using the Lambda matrices, with all the quantities connected using the "line up the indices" rule.

12.2 Automatic conservation of source

In our discussion of conservation laws, we noted that the equation of charge continuity,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \ , \tag{12.23}$$

has a covariant formulation

$$\partial_\alpha J^\alpha = 0 \ . \tag{12.24}$$

Let's revisit this in the context of our covariant formulation of Maxwell equations: taking a derivative of $\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$, we have

$$\partial_\alpha \partial_\beta F^{\alpha\beta} = \mu_0 \partial_\alpha J^\alpha \ . \tag{12.25}$$

The right-hand side of this is zero by virtue of charge continuity. What about the left-hand side? Let's look at it carefully:

$$\begin{aligned} \partial_\alpha \partial_\beta F^{\alpha\beta} &= -\partial_\alpha \partial_\beta F^{\beta\alpha} && \text{(Antisymmetry of } F^{\alpha\beta}\text{)} \\ &= -\partial_\beta \partial_\alpha F^{\beta\alpha} && \text{(Symmetry of } \partial_\alpha\partial_\beta\text{)} \\ &= -\partial_\alpha \partial_\beta F^{\alpha\beta} && \text{(Relabeling of dummy indices)} \end{aligned} \tag{12.26}$$

Comparing the first line with the last we see we again have a situation where the quantity in question is equal to the negative of itself; this is another example of the situation of a symmetric object (in this case, the pair of derivatives $\partial_\alpha\partial_\beta$) contracted onto an antisymmetric one ($F^{\alpha\beta}$). We must have

$$\partial_\alpha \partial_\beta F^{\alpha\beta} = 0 \ . \tag{12.27}$$

This little calculation reveals a very important point: theories of physics in which some source yields a field typically are governed by a set of field equations whose heuristic structure is of the form

$$\text{(Derivatives)(Fields)} = \text{(Source)} \ . \tag{12.28}$$

Sources are never unconstrained; they arise from physical matter, and so respect conservation laws. We can write those conservation laws in the form

$$(\text{Other derivatives})(\text{Source}) = 0\ . \tag{12.29}$$

For this to hold up, we really need to have the mathematical structure which holds our fields respect the rule that

$$(\text{Other derivatives})(\text{Derivatives})(\text{Fields}) = 0\ . \tag{12.30}$$

Although we didn't explicitly set out to make our Faraday tensor fit into this framework, it turns out that it does. This becomes an important point to bear in mind as we think about other kinds of interactions that we might want to fit into a relativistic framework.

12.3 Field invariants

Lorentz transformation act on free indices. Any quantity with no free indices is thus invariant under Lorentz transformations; this is why the scalar product between two 4-vectors, $a^\mu b_\mu$, always yields a Lorentz invariant.

Can we make invariants out of tensors? Certainly! — we just have to combine things, using the metric to lower (or raise) indices, such that there are no free indices for the Lorentz transformation matrix to affect.

Perhaps the simplest one we can construct is called the *trace*. In linear algebra, the trace of a matrix is the sum of its diagonal entries. When we are dealing with tensors, we make this a little more formal: we sum over the indices with one upstairs, and one downstairs. Let's look at this for the Faraday tensor:

$$F^\mu{}_\mu = F^{\alpha\mu}\eta_{\mu\alpha}\ . \tag{12.31}$$

This is a quantity whose values all Lorentz frames agree on. Unfortunately, in this case, it doesn't turn out to be very interesting: using the Faraday tensor $F^{\alpha\beta}$ we've listed above and combining with $\eta_{\mu\alpha} = \mathrm{diag}(-1, 1, 1, 1)$, we get

$$F^\mu{}_\mu = 0 + 0 + 0 + 0 = 0\ . \tag{12.32}$$

The number zero is indeed a Lorentz invariant, but we don't learn anything useful from doing this analysis. (We get the exact same result if we evaluate $G^\mu{}_\mu$.)

We can make others Lorentz invariants by combining the Faraday tensor with itself. Let's look at

$$F^{\alpha\beta}F_{\alpha\beta} = F^{\alpha\beta}F^{\mu\nu}\eta_{\alpha\mu}\eta_{\beta\nu}\ . \tag{12.33}$$

With a little bit of effort, you should be able to show that the Faraday tensor with all indices in the downstairs position is represented by the matrix

$$F_{\alpha\beta} \doteq \begin{pmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ E^x/c & 0 & B^z & -B^y \\ E^y/c & -B^z & 0 & B^x \\ E^z/c & B^y & -B^x & 0 \end{pmatrix}\ ; \tag{12.34}$$

i.e., both row 0 and column 0 are multiplied by negative 1 versus $F^{\alpha\beta}$; cf. Eq. (12.3). Using this, it is straightforward to show that

$$\begin{aligned} F^{\alpha\beta}F_{\alpha\beta} &= 2\left[(B^x)^2 + (B^y)^2 + (B^z)^2 - (E^x/c)^2 - (E^y/c)^2 - (E^z/c)^2\right] \\ &= 2\left[\mathbf{B}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{E}/c^2\right] \ . \end{aligned} \tag{12.35}$$

In other words, *the quantity* $|\mathbf{B}|^2 - |\mathbf{E}|^2/c^2$ *is the same to all Lorentz observers.* This could in principle be deduced by careful study of the Lorentz transformed fields that we derived in the previous lecture, but it follows very simply and easily from the fact that $F^{\alpha\beta}F_{\alpha\beta}$ must be a Lorentz invariant.

There are two other Lorentz invariants we can form from the field tensors. One of them, $G^{\alpha\beta}G_{\alpha\beta}$, is identical to $F^{\alpha\beta}F_{\alpha\beta}$ except for the overall sign, so it yields no new information. But the other one is more interesting:

$$\begin{aligned} F^{\alpha\beta}G_{\alpha\beta} &= 4\left(B^xE^x/c + B^yE^y/c + B^zE^z/c\right) \\ &= 4\mathbf{B}\cdot\mathbf{E} \ . \end{aligned} \tag{12.36}$$

All observers agree on the 3-dimensional dot product of $\mathbf{E}$ *and* $\mathbf{B}$*.* Again, this could have been deduced directly from the fields, but doing with the field tensors is far simpler and more straightforward.

12.4 Potentials and gauge freedom (CAUTION: somewhat advanced material)

[NOTE: I will occasionally want to discuss material that is a bit more advanced than, strictly speaking, we intend for 8.033. When I do this, I will use a "CAUTION" flag as I've written in this section heading. Students who wish to do so can skip over these sections without penalty. Some of this material is likely to fit in better after you have taken additional coursework. For example, this present section is probably best for students who either discussed gauge freedom in their 1st-year E&M class (which doesn't happen for all students), or who have taken 8.07.]

12.4.1 A covariant formulation of electromagnetic potentials

We began our discussion of a covariant formulation of electrodynamics by noting that we cannot "fit" the 6 functions which describe electric and magnetic fields into a 4-vector. A few of you may have wondered: what about the potentials? In freshman electricity and magnetism, we learn that electric fields can be written as the gradient of a scalar potential, and the magnetic field as the curl of a vector potential; in more advanced presentations, we learn that the electric field in situations with time-varying magnetic fields has a contribution from the time-derivative of the vector potential:

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \ , \qquad \mathbf{B} = \nabla\times\mathbf{A} \ . \tag{12.37}$$

One scalar potential, 3 components of vector potential ... this looks tailor-made to fit into a 4-vector! The potentials ϕ and $\mathbf{A}$ have different dimensions, so to make this work we again need to introduce a factor of c. Doing so, we define the 4-potential $\vec{A} = A^\mu \vec{e}_\mu$, whose components are given by

$$A^\mu \doteq \begin{pmatrix} \phi/c \\ A^x \\ A^y \\ A^z \end{pmatrix} . \tag{12.38}$$

We know that $F^{\alpha\beta}$ is antisymmetric, and the fields are built by taking derivatives of the potentials. So let's make an antisymmetric combination of derivatives of fields:

$$X^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha . \tag{12.39}$$

Notice that we are using the "upstairs" partial derivative, $\partial^\alpha = \partial/\partial x_\alpha$. We do this so that we can create tensor components whose indices are all raised, guaranteeing that they have the correct antisymmetry. Recall from Lecture 9 that $x_\alpha \equiv \eta_{\alpha\beta} x^\beta$, and so the components of ∂^α are nearly identical to those of ∂_α. The critical difference is that the zero component has the opposite sign: $\partial^0 = -\partial_0 = -(1/c)\partial/\partial t$.

Let's go through some of the components of $X^{\alpha\beta}$. We can skip X^{00}, X^{11}, X^{22}, X^{33} — the form of Eq. (12.39) guarantees that they are zero. Let's move across row 0:

$$\begin{aligned} X^{01} &= \partial^0 A^1 - \partial^1 A^0 \\ &= -\frac{1}{c}\frac{\partial A^x}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial x} \\ &= E^x/c . \end{aligned} \tag{12.40}$$

Comparing with Eq. (12.3), we see that $X^{01} = F^{01}$. We likewise quickly find that $X^{02} = F^{02}$, and $X^{03} = F^{03}$.

Let's move across row 1. We can skip X^{10} — it will be $-X^{01}$, quickly showing that $X^{10} = F^{10}$. Moving to the first component that is new,

$$\begin{aligned} X^{12} &= \partial^1 A^2 - \partial^2 A^1 \\ &= \frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y} \\ &= (\nabla \times \mathbf{A})^z \\ &= B^z . \end{aligned} \tag{12.41}$$

Comparing with Eq. (12.3), we see that $X^{12} = F^{12}$. By a similar set of calculations, we quickly show that $X^{13} = F^{13}$, and that $X^{23} = F^{23}$. Thanks to the antisymmetry, we are done, and conclude that

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha . \tag{12.42}$$

12.4.2 Gauge freedom

One of the things we learn in electrodynamics classes is that we have some freedom to adjust the form of the potentials, as long as these adjustments have no impact on the fields; after all, it is the fields that exert forces and that are directly measurable. In a particular Lorentz frame, the form that this takes is that we imagine there exists some scalar function λ, which we will call the "gauge generator." It is not difficult to show that if we adjust the potentials as follows,

$$\phi' = \phi - \frac{\partial\lambda}{\partial t}\,, \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda\,, \tag{12.43}$$

then the fields **E** and **B** are unchanged. We prove this by simply computing the fields using ϕ' and $\mathbf{A}'$ rather than ϕ and $\mathbf{A}$:

$$\begin{aligned}
\mathbf{E}' &= -\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t} \\
&= -\nabla\phi + \nabla\frac{\partial\lambda}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial}{\partial t}\nabla\lambda \\
&= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\
&= \mathbf{E}\,; \qquad (12.44)\\
\mathbf{B}' &= \nabla\times\mathbf{A}' \\
&= \nabla\times\mathbf{A} + \nabla\times\nabla\lambda \\
&= \nabla\times\mathbf{A} \\
&= \mathbf{B}\,. \qquad (12.45)
\end{aligned}$$

In the proof for **E**, we used the fact that partial derivatives commute to see that

$$\nabla\frac{\partial\lambda}{\partial t} - \frac{\partial}{\partial t}\nabla\lambda = 0\,; \tag{12.46}$$

for **B**, we used the fact that the curl of the gradient of any scalar function is zero.

The way we bring gauge freedom into the covariant framework is quite simple: we set

$$(A')^\alpha = A^\alpha + \partial^\alpha\lambda\,. \tag{12.47}$$

(We use the somewhat cumbersome notation $(A')^\alpha$ to denote the shifted potential. If the prime were outside the parentheses, it might look like it is attached to the index, and would look like indices corresponding to a different Lorentz frame.) With this, it is simple to see that the Faraday tensor is unchanged:

$$\begin{aligned}
(F')^{\alpha\beta} &= \partial^\alpha (A')^\beta - \partial^\beta (A')^\alpha \\
&= \partial^\alpha A^\beta + \partial^\alpha\partial^\beta\lambda - \partial^\beta A^\alpha - \partial^\beta\partial^\alpha\lambda \\
&= \partial^\alpha A^\beta - \partial^\beta A^\alpha \\
&= F^{\alpha\beta}\,. \qquad (12.48)
\end{aligned}$$

12.4.3 An example application of gauge freedom

If you've never encountered gauge transformations before, you might wonder why we might want to change from one gauge to another. If both gauges give the same fields, and the fields are things that ultimately act on charges and currents, then who cares? What good comes from messing around with this detail?

To see an example of why this can quite useful, let's look at the sourced Maxwell equation, but written in terms of the potential:

$$\partial_\beta F^{\alpha\beta} = \partial_\beta\partial^\alpha A^\beta - \partial_\beta\partial^\beta A^\alpha = \mu_0 J^\alpha\,. \tag{12.49}$$

Because partial derivatives commute, we can swap the order of the derivatives in the first term involving the potential. And, we recognize the combination of derivatives in the second term as the invariant wave operator. The sourced Maxwell equation can thus be rewritten

$$\Box A^\alpha - \partial^\alpha \left(\partial_\beta A^\beta\right) = -\mu_0 J^\alpha \; . \tag{12.50}$$

Equations of the form

$$\Box\,(\text{Function}) = (\text{Source}) \tag{12.51}$$

are particularly lovely in physics — powerful computational techniques make it possible to solve such equations. Unfortunately, the form we've got, (12.50) is not *quite* in that form: it's skewed a bit by the "extra" term $\partial^\alpha(\partial_\beta A^\beta)$. If only we could get rid of it.

Gauge freedom to the rescue. Suppose we change gauge, putting

$$A^\beta_{\rm new} = A^\beta_{\rm old} + \partial^\beta \lambda \; . \tag{12.52}$$

The term which is makes Eq. (12.50) not quite "lovely" for us then involves

$$\partial_\beta A^\beta_{\rm new} = \partial_\beta A^\beta_{\rm old} - \partial_\beta \partial^\beta \lambda = \partial_\beta A^\beta_{\rm old} - \Box\lambda \; . \tag{12.53}$$

If we choose our gauge generator such that

$$\Box\lambda = \partial_\beta A^\beta_{\rm old} \; , \tag{12.54}$$

then the offending term vanishes: we then have

$$\partial_\beta A^\beta_{\rm new} = \partial_\beta A^\beta_{\rm old} - \Box\lambda = 0 \; . \tag{12.55}$$

We can in fact always find a gauge generator λ which satisfies Eq. (12.54); because of this, we can just assume that we have done this analysis, and we use the potential in this new gauge. The sourced Maxwell equation then becomes (dropping the "new" subscript)

$$\Box A^\alpha = -\mu_0 J^\alpha \; . \tag{12.56}$$

When the potential satisfies Eq. (12.56), we say that is in *Lorentz gauge*. This gauge is particularly useful for studies of electromagnetic radiation, since the equation governing the potential is nothing more than a wave equation with a source.

Other gauges exist, and can be really useful in particular reference frames. Such gauges tend not to be "nice" in covariant formulation, though, since they are designed to work only in some frame.