Recap:
- Reviewed stress-energy tensor. Shown that it can be used to give a covariant formulation of energy and momentum conservation:

$$\sum_{\alpha} \partial_{\alpha} T^{\alpha \beta} = 0$$

- Argued that gravity means we can no longer have "global" Lorentz frames. "Global" Lorentz frame means using the metric $g^{\alpha \beta} = \text{diag} (-1, 1, 1, 1)$ everywhere everywhere; if we move away from this, the spacetime metric becomes $g_{\alpha \beta}$, a tensor whose components are functions of position in space and time.

- Guidelines for moving forward provided by the principle of equivalence:

  Over sufficiently small regions, the notion of freely falling bodies due to gravity cannot be distinguished from uniform acceleration.

  In sufficiently small regions of spacetime, the laws of physics in a freely falling frame reduce to those of special relativity.
Two big tasks:
1. Understand how spacetimes grow arise given an arrangement of stress energy \( T_{\mu\nu} \).
2. Understand how, given a spacetime, matter moves around, i.e., how to build geodesics.

"Matter tells spacetime how to curve; spacetime tells matter how to move." — John A. Wheeler.

Item 1: Will do in a somewhat schematic manner in next lecture. Doing this with rigor beyond the scope of 8.033!

Item 2: "Easily" done.

Key idea: Develop your intuition in freely falling frame (FFF). How do bodies move there? Simple: If unaccelerated, they must follow geodesics.

Equivalence principle tells us that if we can formulate the motion in the FFF, we've formulated motion in all frames:

- Motion in a general spacetime is described by a geodesic if the body feels no force (aside from gravity).
Geometry: Previously, we examined geodesics in the following way:

\[ \text{CST} = \int_A^B \sqrt{-g_{\mu \nu} \, dx^\mu \, dx^\nu} \quad \text{Accomplished proper time for a time-like observer} \]

We then chose to single out the coordinate \( t \) as special, and wrote

\[ \Delta \tau = \int_A^B \left[ 1 - \frac{1}{c^2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \right]^{1/2} \, dt \]

where \( \dot{x} = dx/dt \), etc. Defining \( f = \left[ 1 - \frac{1}{c^2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \right]^{1/2} \), we apply Euler equations,

\[ \frac{\partial f}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial f}{\partial q} = 0 \]

and find straight line solutions:

\[ x = \dot{x} t + x_0 \quad \text{etc for } y, z. \]

We want to redo this, fixing up two "flaws".

1. Don't want to pick at time as a special coordinate: picks at some frame as special.
2. We want to handle a spacetime metric that is not constant.
Dealing with the second point isn't so hard - we've mentioned this issue in passing by virtue of the fact that we can do special relativity in "non-inertial" coordinates:

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \quad \text{"Inertial"} \]
\[ = -c^2 dt^2 + dr^2 + r^2 d\phi^2 + dz^2 \quad \text{"Cylindrical"} \]
\[ = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad \text{"Spherical"} \]

We will reserve the symbol \( g_{\alpha \beta} = \text{diag} \{-1, 1, 1, 1\} \) for the metric in inertial coordinates. We will use \( g_{\alpha \beta} \) to denote the metric in general:

\[ ds^2 = \sum_{\alpha \beta} g_{\alpha \beta} dx^\alpha dx^\beta \]

**Cylindrical:**

\[ g_{\alpha \beta} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \]

\[ x^0 = ct \quad x^1 = r \]
\[ x^2 = \phi \quad x^3 = z \]

**Spherical:**

\[ g_{\alpha \beta} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{bmatrix} \]

\[ x^0 = ct \quad x^1 = r \]
\[ x^2 = \theta \quad x^3 = \phi \]
Next, we want to formulate rules describing geodesics in a way that does not pick out coordinate t as "special."

Let's imagine that there's some parameter \( \lambda \) that increases monotonically along the trajectory:

\[ \lambda = 0 \quad \lambda = 1 \quad \lambda = 2 \quad \lambda = 3 \quad \lambda = 4 \quad \lambda = 5 \]

Each value of \( \lambda \) maps uniquely to some event that the body passes through on the trajectory.

The accumulated proper time on the trajectory is

\[
\Delta \tau = \frac{1}{c} \int_A^B \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}} \, d\lambda
\]

\[= \frac{1}{c} \int_A^B \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\lambda\]

where \( \dot{x}^\mu = \frac{dx^\mu}{d\lambda} \)

Now, let's extremize this in the normal way, using \( x^\mu \) and \( \dot{x}^\mu \) as our variational quantities.

Actually, before jumping into this, it is useful to massage a bit first.

Define \( f = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \)

So

\[\Delta \tau = \frac{1}{c} \int_A^B (-f)^{1/2} \, d\lambda\]
As before, we imagine that there is some solution that
gives us the extremum, and consider a variation:

\[ x^\alpha (\lambda) = x^\alpha e (\lambda) + \varepsilon \Delta x^\alpha \]

\[ \text{Extremal trajectory} \]

\[ \text{Deviation that} \]
\[ \text{vanishes at endpoints} \]
\[ \text{of integral} \]

Extremum of \( \Delta x \) defined by

\[ \frac{\partial (\Delta x)}{\partial \alpha} = \frac{1}{c} \int_A^B \frac{\partial}{\partial \alpha} \left( \frac{\partial}{\partial \alpha} (-f)^{1/2} \right) \, d\lambda = 0 \]

\[ \Rightarrow \int_A^B \frac{\partial f / \partial \alpha}{[-f]^{1/2}} \, d\lambda = 0 \]

We can now simplify this with a clever choice
for \( \lambda \). Recall we defined \( \lambda \) simply by demanding
that it increase monotonically from event \( A \) to event \( B \).

Perfect choice for this: let \( \lambda \) simply be proper time
experienced by the body!

With this choice, \( -f = -g_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \)

\[ = -\ddot{u} \cdot \ddot{u} = c^2 \]

\[ \Rightarrow \text{the } f \text{ in the denominator becomes a} \]
\[ \text{constant!} \]
Our condition is then
\[ \frac{\partial (\Delta t)}{\partial x} \propto \int_A^B \frac{\partial f}{\partial x} \, dx = 0 \]

This means the integral we want to extremize is simply
\[ S = \int_A^B L \, dt \]

where
\[ L = \frac{1}{2} f = \frac{1}{2} \sum_{\mu, \nu} g_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = \frac{1}{2} \sum_{\mu, \nu} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu \]

is a Lagrangian for relativistic motion! (Really, a Lagrangian per unit mass.)

When we apply Euler to this,
\[ \frac{\partial L}{\partial x^\mu} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^\mu} = 0 \]

we get equations governing the geodesics of the spacetime \( g_{\mu \nu} \).
In all of the examples we will study in 8.033, we will build \( L \) for a given spacetime, and then build Euler-Lagrange equations to describe motion in this spacetime. Example: Newtonian limit in part #7.

However, it is useful to know that we in fact get an interesting result if we leave the metric totally unspecified: 

\[
L = \frac{1}{2} \sum g_{\mu
u} \dot{x}^\mu \dot{x}^\nu
\]

leads to

\[
\frac{d^2 x^\mu}{d\tau^2} + \sum_{\alpha,\beta} \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0
\]

where

\[
\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \sum_{\sigma=0}^{5} g^{\mu\sigma} \left( \partial_\alpha g_{\sigma\beta} + \partial_\sigma g_{\alpha\beta} - \partial_\beta g_{\alpha\sigma} \right)
\]

The quantity \( \Gamma^\mu_{\alpha\beta} \) is called a "connection coefficient" or "Christoffel symbol," and describes the manner in which the basis vectors, \( \hat{e}_\mu \), vary with position. It connects \( \hat{e}_\mu (\vec{x}) \) with \( \hat{e}_\mu (\vec{x} + \Delta \vec{x}) \).
A few points worth noting:

1. If $\delta g_{ab} = 0$, then
   \[
   \frac{d^2 x^a}{dt^2} = 0 \rightarrow \text{A straight line!}
   \]

2. Suppose we defined our trajectory in a slightly different way: As we moved along the trajectory, imagine that we require our tangent to the trajectory at step $N$ be parallel to the tangent at step $N+1$.

   On blackboard, this defines a straight line.

   On a curved surface, this defines a trajectory that is as straight as possible: A geodesic! This is an alternate way to derive the geodesic equation, and gives us another interpretation of its meaning:

   A geodesic defines the trajectory that is as straight as possible given the geometry on which the trajectory resides.

3. We will never use this general form in 8.033!