Recap:
- Geodesics in greater generality: Found by developing Euler-Lagrange equations, \( \frac{\partial L}{\partial x^k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^k} = 0 \), for the Lagrangian \( L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \) (where \( \dot{x}^k = \frac{dx^k}{dt} \)).

- If you do this for a totally general metric, the geodesics given by the equations
  \[
  \frac{d^2 x^\mu}{dt^2} + \sum_{\alpha, \beta} \Gamma^\alpha_{\beta\mu} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0
  \]
  where \( \Gamma^\alpha_{\beta\mu} = \frac{1}{2} \sum_{\delta=0}^{3} g^{\alpha\delta} (\partial_\delta g_{\beta\mu} + \partial_\beta g_{\mu\delta} - \partial_\mu g_{\beta\delta}) \)

- This also defines "parallel transport": The trajectory we get by moving a vector parallel to itself in some direction in spacetime.
Where do our spacetimes come from? Very, very schematic discussion of the Einstein field equation of GR.

Intuition: Consider Newton's field equation:
\[ \nabla^2 \Phi = 4\pi G \rho \]
Two derivatives of potential source = 1 derivative of grav. acceleration = **TIDAL field**

Newton: "Tides" = "Matter density"

Einstein: Matter density \( \rightarrow \) \( T^{\alpha\beta} \). Expect "Tides" = "Stress-energy"

So, how do we make "tides" rigorous?

As we've already discussed, tides are related to a spacetime's curvature. They describe how parallel trajectories in spacetime converge or diverge. So, we just need to quantify this convergence or divergence.
Geodesic 1: \[ \frac{d^2 x^\mu}{dt^2} + \sum_{\nu, \rho} \Gamma^\mu_{\nu \rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} \]

Evaluated on path 1

Geodesic 2: Same thing, but now evaluated on path 2.

Finally, let \( \delta x^\mu = x_2^\mu - x_1^\mu \). Then,

with some effort, we can show that

\[ \frac{D^2 (\delta x^\mu)}{dt^2} = R^\mu_{\alpha \rho \nu} \frac{dx^\alpha}{dt} \frac{dx^\rho}{dt} \delta x^\nu \]

where \( \frac{D}{dt} \) is a "covariant" derivative, one that accounts for the curvature of the metric in a careful way, and where

\[ R^\mu_{\alpha \rho \nu} = \partial_\rho \Gamma^\mu_{\nu \alpha} - \partial_\nu \Gamma^\mu_{\rho \alpha} + \sum_{\delta = 0}^3 \left( \Gamma^\mu_{\beta \delta} \Gamma^\beta_{\nu \alpha} - \Gamma^\mu_{\beta \nu} \Gamma^\beta_{\rho \alpha} \right) \]

is the "Riemann curvature tensor."
A few points to note:

1. It has terms of the form \( e^P \), which is itself of the form \( \Omega \) (metric). Hence,
\[
\text{Curvature} \sim \Omega^2 \text{ (metric)}
\]
- Just like curvature of a curve in 4-D is related to its 2nd derivative.
- Just as tidal force looks like 2nd derivative of potential.

2. It has terms of the form \( P^2 \sim (\Omega \text{metric})^2 \)
   - Non-linear!

3. It has 4 indices — naively, it has
\[
y^4 = 256 \text{ components!}
\]
Symmetries in fact reduce this: Only 20 of those 256 are independent. Still, 20 is a mess to deal with.
Following our earlier logic, we expect our gravity field equation to be of the form

\[(\text{Riemann}) = \lambda \, T_{\mu \nu}\]

where \(\lambda\) is some constant needed to get dimensions right. However, Riemann has 4 indices, stress-energy only 2.

Solution: Use metric to combine two of the indices, called the trace:

\[R_{\mu \nu} = \sum_{\alpha = 0}^{3} \sum_{\beta = 0}^{3} g^{\alpha \beta} R^{\alpha \beta}_{\mu \nu}\]

Connect 1st + 3rd indices.

**Attempt 2:**

\[R_{\mu \nu} = \lambda \, T_{\mu \nu}\]

Does this work?

Not quite. Recall special relativity that \(\sum_{\mu} T^{\mu}_{\text{trace}} = 0\).

A curved spacetime variant also holds:

\[\sum_{\mu = 0}^{3} \nabla_{\mu} T_{\mu \nu} = 0\]

"Covariant" analog of \(\partial_{\nu}\) accounts for curvature of spacetime.
Unfortunately, \( \sum_{\mu=0}^{3} \nabla_\mu R^\mu = 0 \). However, it is not difficult to show that

\[
\sum_{\mu=0}^{3} \nabla_\mu \left( R^\mu - \frac{1}{2} g^\mu_\nu R \right) = 0,
\]

where \( R = \sum_\mu g^\mu_\nu R^\mu \).

Definition: \( G^\mu_\nu = R^\mu_\nu - \frac{1}{2} g^\mu_\nu R \) "Einstein curvature."

Our candidate equation is then

\[
G^\mu_\nu = \kappa T^\mu_\nu
\]

If \( \kappa = \frac{8\pi G}{c^{4}} \), then in an appropriate limit, this reproduces Newtonian gravity:

\[
G^\mu_\nu = \frac{8\pi G}{c^{4}} T^\mu_\nu
\]

The Einstein field equation of general relativity.

Note general structure:

"Big nonlinear coupled mess of metric = Flow of energy and momentum"
Some important solutions:

1. \( g_{00} = -(1 + 2 \Phi) \)
\[ g_{11} = g_{22} = g_{33} = 1 - 2 \Phi \]
\[ x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z \]

\[ \Phi = -\frac{GM}{rc^2} \quad r = \sqrt{x^2+y^2+z^2} \]

Corresponds to a spherical body of mass \( M \) for \( \Phi \ll 1 \) ... reproduces Newtonian gravity.

2. \( g_{00} = -(1 - 2GM/rc^2) \)
\[ g_{11} = (1 - 2GM/rc^2)^{-1} \]
\[ g_{22} = r^2 \quad g_{33} = r^2 \sin^2 \theta \]
\[ x^0 = ct \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi \]

The Schwarzschild metric: Exact solution outside any spherical distribution of mass \( M \).

If body satisfies \( T^{\mu \nu} = 0 \) everywhere except at \( r=0 \), then this represents a non-spining black hole.

"Schwarzschild metric" - Karl Schwarzschild, 1916.
3. \[ g_{00} = -\left(\frac{\Delta - a^2 \sin^2 \Theta}{\Sigma}\right) \quad g_{ii} = \frac{\Sigma}{\Delta} \quad g_{zz} = -\Sigma \]

\[ g_{33} = \left(\frac{(v^2 + a^2)^2 - a^2 \Delta \sin^2 \Theta}{\Sigma}\right) \sin^2 \Theta \]

\[ g_{03} = g_{30} = -\frac{2aM\tilde{r} \sin^2 \Theta}{\Sigma} \quad "Kerr\ metric" \quad 1963 \]

\[ \Delta = v^2 - 2\tilde{M}v + a^2 \]

\[ \Sigma = v^2 + a^2 \cos^2 \Theta \]

\[ \tilde{M} = \frac{GM}{c^2} \quad a = \frac{J}{MC} \]

\[ x^0 = ct \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi \]

Represents spacetime of a black hole with mass \( M \) and spin angular momentum \( J \). "Kerr metric."

4. \[ g_{00} = 1 \quad g_{ii} = a^2 (t) \quad g_{zz} = a^2 (t) v^2 \]

\[ g_{33} = a^2 (t) v^2 \sin^2 \Theta \]

Function \( a(t) \) determined by density \( \delta \), pressure \( P \):

\[ \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \delta}{3c^2} \]

\[ \frac{\ddot{a}}{\dot{a}} = -\frac{4\pi G}{3c^2} (\delta + 3P) \]

Describes universe filled of perfect fluid with density \( \delta \) and pressure \( P \).

"Friedman - Robertson - Walker metric"
Example calculations: Newtonian metric.

Suppose we have an observer who is at rest in the "Newtonian" spacetime: \( \frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0 \). What is \( dt/\Delta t \)?

We still require \( \tilde{u} \cdot \tilde{u} = -c^2 \). Why? If we go into a freely falling frame, the calculation is identical to special relativity. Invariance of inner product tells us that the result holds in general:

\[
\tilde{u} \cdot \tilde{u} = \sum g_{\alpha\beta} u^\alpha u^\beta = - (1 + 2\Phi) c^2 \left( \frac{dt}{\Delta t} \right)^2 + 0 = -c^2
\]

\[
\Rightarrow \frac{dt}{\Delta t} = (1 + 2\Phi)^{-1/2} \approx (1 - \Phi) \quad (\text{since } \Phi \ll 1)
\]

\[
\approx 1 + \frac{GM}{\sqrt{c^2}}.
\]

Consider observers at two different heights:

\[
\frac{(dt/\Delta t)^2}{(dt/\Delta t_N)^2} = \frac{1 + GM/\sqrt{c^2}}{1 + GM/\sqrt{c^2}^2} \approx 1 + \frac{GM}{c^2} \left( \frac{1}{\sqrt{c^2}} - \frac{1}{c^2} \right)
\]

\[
= 1 - \frac{\Delta \Phi_N}{c^2}
\]

Same result we obtained for ticking of clocks by considering light redshift.
These metrics give us a relativistic way to compute the effect of gravity. Involves a big change in philosophy:

Newton: Gravity is a force: $E_g = -\frac{GMm}{r^2} \quad r = \text{m}, g$.

Einstein: Bodies follow geodesics of spacetime. If spacetime is "weakly curved," then the trajectory for slow motions is identical to what the Newtonian gravitational force predicts.

Score card:

1. Reproduces Newton when $\frac{GM}{rc^2} \gg 1$, and for non-relativistic motions ($\frac{r}{c} \ll 1$).
   - If it had not, it would not have been a viable theory!

2. Predicts clocks run at different rates, and redshifting of light in gravity.
   - Measurable every day. Key component of high precision position measurements.

   - Weak, but measurable effect in solar system.
   - Eddington measured it in 1919. Can now measure with part per million accuracy!
   - We now exploit this effect: Measure light bending, infer how much mass/energy is present: Gravitational/ lensing.
4. Corrections to Newtonian motion

On page 1, you examined the limit $\Phi \to 0$ and $l(x) \to 0$. Geodesics in this limit reproduce the Newtonian force law:

$$\frac{d^2 x}{dt^2} = -\frac{GM}{r^3} x$$

Newton's gravity predicts orbits that are closed ellipses:

![Elliptical orbit](image)

Ellipticity highly exaggerated...

Observations the centuries show that the ellipses don't quite close - instead they precess. The axes slowly rotate around the sun:

![Elliptical precession](image)

Most of this effect is due to interactions with other planets. However, there is a small residual which could not be accounted for.

Largest residual for Mercury: Ellipse precesses at 43 arcseconds per century.

Note: 1 degree = 3600 arcseconds.
Repeat exercise from part #7, but now keep 1st corrections to this leading solution (i.e. don't discard terms of order \( v^2/c^2 \)).

Result:

\[
\frac{d^2 x}{dt^2} = -\frac{GMx}{r^3} - \frac{GM}{r^2} \frac{v_1^2}{c^2} x + \frac{4GM(x, y)}{c^2 r^3}
\]

This force gives an ellipse that precesses. Rate of precession we find \( \dot{\Phi} \) for Mercury is

\[
\frac{d\Phi}{dt} = \frac{GM0}{a(1-e^2)Pc^2}
\]

\( M_0 \) = mass of sun = \( 1.99 \times 10^{30} \) kg

\( a \) = semi-major axis of Mercury's orbit

\( = 57.9 \times 10^6 \) km

\( e \) = eccentricity of Mercury's orbit

\( = 0.2 \)

\( P \) = Period of Mercury's orbit

\( = 88 \) days

\[
\rightarrow \frac{d\Phi}{dt} = 42.9 \text{ arcseconds per century.}
\]