The value of $\theta$ and the behavior of $\alpha(t)$ must be determined from the Einstein field equations, which we will do soon. As a setup, it is useful to examine how light and matter behave in this spacetime.

Begin by asking what happens to observers at rest:

$$u^t = c, \quad u^\nu = u^\theta = u^\phi = 0.$$

Examine geodesics: Find that they remain fixed at coordinate $(\vec{r}, \theta, \phi)$.

However, as they are fixed there, the proper separation of different observers changes as $a(t)$ evolves. These observers "co-move" as the universe's geometry changes—they are "comoving" observers.

Next, examine light—Crucial, since we use it to measure and understand our universe.

For simplicity, focus on $d\epsilon = 0$. (Calculation can be generalized to $d\epsilon = \pm 1$, but it is messy.)

Imagine that light is emitted at some time $t$, and received by an observer at some later time $t'$. It's enough to consider light that moves radially, so we'll put $\vec{p} = (p^t, p^\nu, 0, 0)$. 
Goal: Compare energy of light when it is emitted to the energy when it is received. To do this, we imagine a comoving observer at emission and reception:
\[ E_{\text{emit}} = -\vec{p}_{\text{emit}} \cdot \vec{v}_{\text{emit}} = \vec{p}_{\text{emit}}^+ c \]

Since \( \vec{v} = -c \) for a comoving observer.

Now, let's propagate this across spacetime as a radial geodesic and see what energy it gets at \( t = t_0 \).

Two rules regarding the light:

1. Light-like trajectory, so \( \vec{p} \cdot \vec{v} = 0 \):
   \[ -(p^+)^2 + a^2 (t_0) (p^-)^2 = 0 \Rightarrow p^- = p^+ / a(t_0) \]

2. Geodesic, so we extremize

\[ L = \frac{1}{2} \sum g_{\mu \nu} \frac{dx^\mu}{dx^\lambda} \frac{dx^\nu}{dx^\lambda} \]

\[ = -\frac{c^2}{2} \left( \frac{dt}{dx} \right)^2 + \frac{a^2 (t_0)}{2} \left( \frac{dx}{dx} \right)^2 \]
Let's focus on the $x^0 = ct$ component:

$$\frac{\partial L}{\partial x^0} = \frac{1}{c} \frac{\partial L}{\partial t} = \frac{1}{c} a \dot{a} \left( \rho^2 \right)_t, \quad \ddot{a} = \frac{da}{dt}$$

$$\frac{\partial L}{\partial \frac{dx^0}{dx}} = -c \frac{dt}{dx} = -\rho_t$$

$$\frac{d}{dx} \left[ \frac{\partial L}{\partial \frac{dx^0}{dx}} \right] = -\frac{dp^t}{dx}$$

$$\frac{\partial L}{\partial x^0} - \frac{d}{dx} \frac{\partial L}{\partial \frac{dx^0}{dx}} = 0 \quad \rightarrow$$

$$\frac{a}{c} \ddot{a} \left( \rho^2 \right)_t + \frac{dp^t}{dx} = 0$$

$$-\frac{1}{c} \dot{a} \left( \rho^2 \right)_t + \frac{dp^t}{dx} = 0$$

But, $\dot{a} \rho_t = \left( \frac{da}{dt} \right) c \frac{dt}{dx} = c \frac{da}{dx}$

Our equation becomes

$$\frac{da}{dx} \rho_t + \frac{dp^t}{dx} = 0$$

$$-\frac{da}{dx} = -\frac{dp^t}{dx} / \rho_t$$

Integrate both sides from $\lambda = \lambda_e$ (corresponding to $t = t_e$) to $\lambda = \lambda_{re}$ (corresponding to $t = t_{re}$):

$$\ln \left[ \frac{\rho_t (t_e)}{\rho_t (t_{re})} \right] = -\ln \left[ \frac{a (t_{re})}{a (t_e)} \right]$$
or \[ \frac{pt(t_e)}{pt(t_e)} = \frac{a(t_e)}{a(t_e)} \]

We measure with a comoving observer, and so \[ E_{\text{rec}} = pt(t_e)c, \] which gives us

\[ \frac{E_{\text{rec}}}{E_{\text{emit}}} = \frac{a(t_e)}{a(t_e)} \]

In words, this means that the way energy associated with light behaves gives us a way to directly probe the scale factor of the universe. This is **HUBBLE** important! How do we use it?

Excited atoms and molecules emit light with distinct spectral lines:

\[ \lambda_1, \lambda_2, \lambda_3, \lambda_4: \text{Very well characterized by lab measurements.} \]

Each wavelength corresponds to a distinct energy level: \[ E = h\nu = \frac{hc}{\lambda}. \] When we measure it, the light has been redshifted due to the scale factor's evolution:
\[
\frac{\lambda_1'}{\lambda_1} = \frac{\lambda_2'}{\lambda_2} = \frac{\lambda_3'}{\lambda_3} = \frac{\lambda_4'}{\lambda_4} = \frac{a(t_e)}{a(t_0)} = 1 + z,
\]

where \( z = \text{"the redshift"} \)

Do this for many sources, build a map of how the scale factor evolves.

Next: How do we get the evolution of the scale factor as a function of time, and how does this relate to the matter content of the universe?