Recap:

- Forces in S.R. accounted for in two different ways. 4-force: simple 4-vector, gives us a geometric spacetime object:

$$\vec{F} = \frac{d\vec{p}}{d\tau} = m\vec{a}$$

τ = proper time of body experiencing the force.

Unfortunately, this does not tie into measurements very well: We measure change in momentum per unit *our own time*, not per unit time of the body experiencing the force. So we also need 3-force:

$$\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt}$$

$\underset{\sim}{p}$, t tied to a particular frame.

- Electric & magnetic fields "too big" for 4-vectors. Need a larger mathematical structure to accommodate the 6 components of $\underset{\sim}{E}$ & $\underset{\sim}{B}$: leads us to the discussion of <u>tensors</u>.

Tensors

A "second-rank tensor" is a geometric object associated with 2 directions in spacetime:

$$\overleftrightarrow{A} = \sum_{\substack{\mu=0\\ \nu=0}}^{3,3} A^{\mu\nu}\, \vec{e}_\mu \otimes \vec{e}_\nu$$

Double-headed arrow: Better notation than the clumsy "fat letter" I used last time!

Components of $\overleftrightarrow{A}$ in some frame

"outer product" or "tensor product"

Unit vector in that frame.

Just as a 4-vector is a geometric object whose properties all observers agree on, so is a tensor. Because of this, we can easily deduce the rules for transforming its components:

$$\overleftrightarrow{A} = \sum_{\mu,\nu=0}^{3} A^{\mu\nu}\, \vec{e}_\mu \otimes \vec{e}_\nu$$

$$= \sum_{\alpha',\beta'=0}^{3} A^{\alpha'\beta'}\, \vec{e}_{\alpha'} \otimes \vec{e}_{\beta'}$$

We already know $\vec{e}_{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\mu}{}_{\alpha'}\, \vec{e}_\mu$

The requirement that $\overleftrightarrow{A}$ be an invariant object leads us to

$$A^{\alpha'\beta'} = \sum_{\mu,\nu=0}^{3} \Lambda^{\alpha'}{}_{\mu}\, \Lambda^{\beta'}{}_{\nu}\, A^{\mu\nu}$$

(Incidentally, a "1^{ST} rank" tensor is an object associated with just 1 spacetime direction … i.e., it's a 4-vector! And, a "0^{TH} rank" tensor is an object with NO associated … i.e., a scalar!)

We need to introduce a few additional tools in order to start using tensors to organize the electric and magnetic fields. Very important tool is a geometric object called the metric:

$$\overleftrightarrow{\eta} = \eta^{\mu\nu} \, \vec{e}_\mu \otimes \vec{e}_\nu$$

The components of this tensor are

$$\eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \mathrm{diag}(-1, 1, 1, 1).$$

Strictly speaking this the "inverse" metric … more on this later.

This metric tensor has a very special property. Suppose we transform to a new frame. The components in that frame are given by

$$\eta^{\alpha'\beta'} = \sum_{\mu,\nu=0}^{3} \Lambda^{\alpha'}{}_{\mu} \Lambda^{\beta'}{}_{\nu}\, \eta^{\mu\nu}$$

Do the math; you'll find

$$\eta^{\alpha'\beta'} = \mathrm{diag}(-1,1,1,1)$$

→ The components of the metric tensor are the same in all Lorentz frames.

Utility of this object: First, we need to invert it:

$\eta_{\mu\nu}$ is defined by

$$\sum_{\alpha=0}^{3} \eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^{\mu}{}_{\nu}$$

The result is trivial! - the components $\eta_{\mu\nu}$ are given by the matrix

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1).$$

Now, for "fun", let's imagine taking this and "contracting" its two indices with the indices of a pair of 4-vectors:

$$\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} A^\mu B^\nu$$

$$= \eta_{00} A^0 B^0 + \eta_{11} A^1 B^1 + \eta_{22} A^2 B^2 + \eta_{33} A^3 B^3$$

$$= -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

$$\equiv \vec{A} \cdot \vec{B}$$

→ The metric gives us a mathematical toolkit that lets us take the dot product of two 4-vectors!

Your reaction to this could very well be "So what? I already know how to do this!" Two reasons why this is interesting:

1. Recall that the interval $\Delta s^2 = \Delta\vec{x} \cdot \Delta\vec{x}$ generalizes the Pythagorean theorem to spacetime. In particular, it describes a notion of "distance" between events when spacetime is "flat." We will soon want to go beyond "flat" spacetime, and need to have a dependable tool that we can use to define the distance between events. The way we'll do this will be to let the metric tensor become a function that varies with position: $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(t, x, y, z)$. Everything else then carries over.

2. More immediately relevant for us, the metric lets us combine vectors and tensors in a sensible way:

Example: We have a tensor $\overleftrightarrow{A}$ and a 4-vector $\vec{B}$. Using the metric, we combine them to make a new 4-vector, $\vec{D}$:

$$\vec{D} = \sum_{\mu=0}^{3} D^\mu \vec{e}_\mu \,, \quad \text{where}$$

$$D^\mu = \sum_{\alpha,\beta=0}^{3} A^{\mu\alpha} \eta_{\alpha\beta} B^\beta$$

→ Kind of like a "dot product" between tensors and vectors.

One thing to be careful about: This generalized dot product is ambiguous! What we get depends on which index on the tensor we sum over.

Example 2: Take the same tensor, same 4-vector. Now, combine them as follows:

$$F^\mu = \sum_{\alpha,\beta=0}^{3} A^{\alpha\mu} \eta_{\alpha\beta} B^\beta$$

$$\vec{F} = \sum_{\mu=0}^{3} F^\mu \vec{e}_\mu$$

Notice sum is over 1^{ST} index rather than 2^{ND} index. $\vec{E} = \vec{F}$ only if $A^{\alpha\beta} = A^{\beta\alpha}$. Index notation makes this totally unambiguous!

We use the metric so often when working with tensors that writing it out gets annoying. So, there's a shorthand we've introduced:

$$A_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} A^\nu$$

"Lowering" an index. Using this, we can write the dot product as

$$\vec{A} \cdot \vec{B} = \sum_{\mu=0}^{3} A_\mu B^\mu = \sum_{\mu=0}^{3} A^\mu B_\mu$$

And, we can write the combination of tensor and vector ~~m~~ with this shorthand:

$$D^\mu = \sum_{\alpha=0}^{3} A^{\mu\alpha} B_\alpha$$

$$F^\mu = \sum_{\alpha=0}^{3} A^{\alpha\mu} B_\alpha$$

The action of lowering is really simple: Since $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, lowering a 4vector's index just flips the sign of its timelike component:

$$A_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} A^\nu \rightarrow \begin{aligned} A_x &= A^x \\ A_y &= A^y \\ A_z &= A^z \end{aligned}$$

But $$A_t = -A^t$$

Why are we introducing all this mathematical structure?

This seems like a lot notation ... do we really need it?

Perhaps we don't need it ... but we've learned to love it because it makes it manifestly obvious how quantities transform between reference frames.

→ Any object with an index in the "upstairs" position transforms just like the component of a 4-vector. ("Contravariant")

→ Any object with an index in the "downstairs" position transforms just like the unit 4-vector ("Covariant")

→ Any object with no indices is a scalar and has the same value in all reference frames.

Consider dot product: $\sum_{\mu=0}^{3} A^\mu A_\mu \equiv a$

When we change into a new reference frame,

$$A^{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\alpha'}{}_{\mu} A^\mu$$

$$A_{\alpha'} = \sum_{\nu=0}^{3} \Lambda^{\nu}{}_{\alpha'} A_\nu$$

Now, examine dot product in this new frame:

$$\sum_{\alpha'=0}^{3} A^{\alpha'} A_{\alpha'} = \sum_{\alpha'=0}^{3} \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \Lambda^{\alpha'}{}_{\mu} \Lambda^{\nu}{}_{\alpha'} A^{\mu} A_{\nu}$$

$$= \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \sum_{\alpha'=0}^{3} \left(\Lambda^{\alpha'}{}_{\mu} \Lambda^{\nu}{}_{\alpha'}\right) A^{\mu} A_{\nu}$$

$$= \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \delta_{\mu}{}^{\nu} A^{\mu} A_{\nu}$$

$$= \sum_{\mu=0}^{3} A^{\mu} A_{\mu} = a \rightarrow$$ Lorentz invariance!

→ When we "contract" over indices, it is as though those indices are not there!

This is the power of the index notation: It organizes all the quantities we use to describe the physics in a way that makes its transformation properties clear.

In particular, can see a simple way to make Lorentz invariant scalars: just make sure that there are no leftover "free" indices. They all need to be contracted away.