Recap:

- Lots of notation! Introduced the "metric" tensor, whose components have the same value in all reference frames: $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$

- Metric gives us a formal tool for writing the dot product of two vectors:

$$\vec{A} \cdot \vec{B} = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} A^\mu A^\nu$$

Kind of trivial for now... but rule continues to hold when we examine non-flat spacetimes, and this allows us to describe a dot product involving tensors:

$$D^\mu = \sum_{\alpha,\beta=0}^{3} A^{\mu\alpha} \eta_{\alpha\beta} B^\beta$$

$$\equiv \sum_{\alpha=0}^{3} A^{\mu\alpha} B_\alpha$$

- Point of all this: Let's us combine geometric objects to make new objects and see, at a glance, how that object transforms between different Lorentz frames.

Let's start applying these ideas to see how we can get the electric and magnetic fields to fit into a special relativistic formulation.

We argued a few lectures ago that the best way to get the 6 components of $\underset{\sim}{E}$ & $\underset{\sim}{B}$ into a single geometric object is to use an antisymmetric tensor:

$$F^{\mu\nu} = - F^{\nu\mu}$$

"Faraday" tensor or "Field" tensor

Now, want to figure out how to get those $\underset{\sim}{E}$ & $\underset{\sim}{B}$ components into $F^{\mu\nu}$.

Goal: Reproduce the Lorentz force law,

$$\frac{d\underset{\sim}{p}}{dt} = q\,\underset{\sim}{v} \times \underset{\sim}{B} + q\,\underset{\sim}{E}$$

but using 4-vectors and tensors.

Our rule should be

- Linear in charge q
  (Note: q is a Lorentz invariant!)
- Linear in the field tensor $F^{\mu\nu}$.

Problem: $F^{\mu\nu}$ is a 2ND-rank tensor (two indices)

... force is a 4-vector (one index).

Need to get rid of one index!

Idea: Contract $F^{\mu\nu}$ with 4-velocity, $u_\mu = \eta_{\mu\nu} u^\nu$.
Need to have 3-velocity anyway for the magnetic force law.

Proposed force law:

$$\frac{dp^\mu}{d\tau} = m a^\mu = q \sum_{\nu=0}^{3} F^{\mu\nu} u_\nu$$

Before doing any calculations, let's check one thing:
We must have 4-velocity orthogonal to 4-acceleration.

$$\vec{a} \cdot \vec{u} = \sum_{\mu=0}^{3} a^\mu u_\mu = 0$$

Does our law respect this? Let's check.

$$\vec{a} \cdot \vec{u} = \frac{q}{m} \sum_{\mu,\nu=0}^{3} F^{\mu\nu} u_\nu u_\mu$$

$$= -\frac{q}{m} \sum_{\mu,\nu=0}^{3} F^{\nu\mu} u_\nu u_\mu \qquad \text{antisymmetry of } F^{\mu\nu}$$

$$= -\frac{q}{m} \sum_{\mu,\nu=0}^{3} F^{\nu\mu} u_\mu u_\nu \qquad \text{symmetry of } u_\nu u_\mu$$

But, quantities on the right-side under the sum on the 1ST & 3RD lines are <u>identical</u>: Because the indices are summed, it <u>does</u> <u>not</u> <u>matter</u> what we label them.

The key thing that matters is that the 1ST index of $F^{\mu\nu}$ is attached to the second 4-velocity, and vice-versa.

So, 1ST and 3RD lines are equal ... but have opposite signs. Only way for this to be true is if their value is zero: $\vec{a} \cdot \vec{u} = 0$

→ This is good! Our proposed force law must be one that obeys this rule.

(This is also an example of a trick often used when manipulating tensors: Whenever an antisymmetric object is totally contracted onto a symmetric object, the result must be zero.)

Now, let's look at this more concretely. Consider the Lorentz force in a frame where the charge is at rest. In this case, the 4-velocity has only one non-zero component:

$$u^t = c \rightarrow u_t = -c.$$

Force law:
$$\frac{dp^\mu}{d\tau} = q F^{\mu\nu} u_\nu$$
$$= -qc\, F^{\mu 0}$$

Since the charge is at rest, $d\tau = dt$. Also, $F^{00} = 0$ due to antisymmetry.

So the force components are

$$\frac{dp^x}{dt} = -qc\, F^{10} \;\longrightarrow\; F^{10} = -E^x/c$$

$$\frac{dp^y}{dt} = -qc\, F^{20} \;\longrightarrow\; F^{20} = -E^y/c$$

$$\frac{dp^z}{dt} = -qc\, F^{30} \;\longrightarrow\; F^{30} = -E^z/c$$

Here are the components of the field tensor so far:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & F^{12} & F^{13} \\ -E^y/c & -F^{12} & 0 & F^{23} \\ -E^z/c & -F^{13} & -F^{23} & 0 \end{bmatrix}$$

To pin down the last components, consider the force on a charge in motion:

$$u_0 = -u^0 = -\gamma c \;,$$

$$u_{x,y,z} = u^{x,y,z} = \frac{d(x,y,z)}{d\tau} = \gamma\, \frac{d(x,y,z)}{dt}$$

$$\frac{dp^x}{d\tau} = q\, F^{1\mu} u_\mu$$

$$= \gamma q\, E^x + q\, F^{12} \gamma \frac{dy}{dt} + q\, F^{13} \gamma \frac{dz}{dt}$$

-or-

$$\frac{dp^x}{dt} = q \left[E^x + F^{12} \frac{dy}{dt} + F^{13} \frac{dz}{dt} \right]$$

To agree with Lorentz force law, should have

$$\frac{dp^x}{dt} = q\left[E^x + B^z \frac{dy}{dt} - B^y \frac{dz}{dt}\right]$$

$$\rightarrow F^{12} = B^z = -F^{21}$$

$$F^{13} = -B^y = -F^{31}$$

Consider other components, deduce that $F^{23} = B^x = -F^{32}$.

So:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & B^z & -B^y \\ -E^y/c & -B^z & 0 & B^x \\ -E^z/c & B^y & -B^x & 0 \end{bmatrix}$$

→ We now have a nice, special-relativity-complete description of the electric & magnetic field!

Rather than treating them as separate fields, in S.R. $\underset{\sim}{E}$ & $\underset{\sim}{B}$ are just different manifestations of a single, unified object.

Note: Lorentz force law has a new component:

$$\frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau} = q F^{0\mu} u_\mu$$

$$\rightarrow \frac{dE}{dt} = q \underset{\sim}{E} \cdot \underset{\sim}{u} \rightarrow \text{work done on charge.}$$

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Let's use this to see how fields transform between frames of reference. Very simple to express law for Faraday tensor:

$$F^{\mu'\nu'} = \sum_{\alpha,\beta=0}^{3} \Lambda^{\mu'}{}_{\alpha} \Lambda^{\nu'}{}_{\beta} F^{\alpha\beta}$$

Take primed frame to move with $\underset{\sim}{v} = v \underset{\sim}{e}_x$ as seen by unprimed frame:

$$\Lambda^{\mu'}{}_{\alpha} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Only non-zero components are

$$\Lambda^{0'}{}_{0} = \gamma \,, \quad \Lambda^{1'}{}_{1} = \gamma \,, \quad \Lambda^{2'}{}_{2} = 1 \,, \quad \Lambda^{3'}{}_{3} = 1$$

$$\Lambda^{1'}{}_{0} = -\gamma\beta = \Lambda^{0'}{}_{1}$$

Using this, let's now examine components of Faraday tensor in the primed frame:

$$F^{0'0'} = \Lambda^{0'}{}_0 \Lambda^{0'}{}_1 \; F^{01} + \Lambda^{0'}{}_1 \Lambda^{0'}{}_1 \; F^{10}$$

$$= 0 \qquad \text{since} \quad F^{01} = -F^{10}$$

→ Can likewise show easily that $F^{1'1'} = F^{2'2'} = F^{3'3'} = 0$

~ An important sanity check!

$$F^{0'1'} = \Lambda^{0'}{}_0 \Lambda^{1'}{}_1 \; F^{01} + \Lambda^{0'}{}_1 \Lambda^{1'}{}_0 \; F^{10}$$
$$= (\gamma^2 - \gamma^2\beta^2) \, F^{01} = F^{01}$$

Translate to $\underset{\sim}{E}$ & $\underset{\sim}{B}$ components, this means

$$\boxed{E^{x'} = E^{x}}$$

$$F^{0'2'} = \Lambda^{0'}{}_0 \Lambda^{2'}{}_2 \; F^{02} + \Lambda^{0'}{}_1 \Lambda^{2'}{}_2 \; F^{12}$$
$$= \gamma \, (F^{02} - \beta F^{12})$$

$$\rightarrow \quad \boxed{E^{y'} = \gamma(E^{y} - vB^{z})}$$

$$F^{0'3'} = \Lambda^{0'}{}_0 \Lambda^{3'}{}_3 \; F^{03} + \Lambda^{0'}{}_1 \Lambda^{3'}{}_3 \; F^{13}$$
$$= \gamma \, (F^{03} - \beta F^{13}) \quad \rightarrow \quad \boxed{E^{z'} = \gamma(E_z + vB^{y})}$$

$$F^{2'3'} = \Lambda^{2'}{}_{2} \Lambda^{3'}{}_{3} F^{23} = F^{23}$$

$$\rightarrow \quad \boxed{B^{x'} = B^{x}}$$

$$F^{1'3'} = \Lambda^{1'}{}_{1} \Lambda^{3'}{}_{3} F^{13} + \Lambda^{1'}{}_{0} \Lambda^{3'}{}_{3} F^{03}$$

$$= \gamma \left(F^{13} - \beta F^{03} \right)$$

$$\rightarrow \quad \boxed{B^{y'} = \gamma \left(B^{y} + \frac{v}{c^2} E^{z} \right)}$$

$$F^{1'2'} = \Lambda^{1'}{}_{1} \Lambda^{2'}{}_{2} F^{12} + \Lambda^{1'}{}_{0} \Lambda^{2'}{}_{2} F^{02}$$

$$= \gamma \left(F^{12} - \beta F^{02} \right)$$

$$\rightarrow \quad \boxed{B^{z'} = \gamma \left(B^{z} - \frac{v}{c^2} E^{y} \right)}$$

These are the laws which govern how electric and magnetic fields transform between Lorentz frames.

These rules for transforming the fields can be repeated for relative motion in any direction. The result is nicely summarized by the following very general formulas:

$$\boxed{\begin{aligned} \underset{\sim}{E}'_{\parallel} &= \underset{\sim}{E}_{\parallel} \qquad & \underset{\sim}{E}'_{\perp} &= \gamma\left(\underset{\sim}{E}_{\perp} + \underset{\sim}{v} \times \underset{\sim}{B}_{\perp}\right) \\ \underset{\sim}{B}'_{\parallel} &= \underset{\sim}{B}_{\parallel} \qquad & \underset{\sim}{B}'_{\perp} &= \gamma\left(\underset{\sim}{B}_{\perp} - \underset{\sim}{v} \times \underset{\sim}{E}_{\perp}/c^2\right) \end{aligned}}$$

$\underset{\sim}{E}_{\parallel}$ means "the component of $\underset{\sim}{E}$ parallel to $\underset{\sim}{v}$", and $\underset{\sim}{E}_{\perp}$ means "the component of $\underset{\sim}{E}$ perpendicular to $\underset{\sim}{v}$".

Likewise for $\underset{\sim}{B}_{\parallel}$ and $\underset{\sim}{B}_{\perp}$.

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