Recap:

- Found a convenient way to write the electromagnetic force law: $\frac{dp^\mu}{d\tau} = q F^{\mu\nu} u_\nu$
  - Gives force on a charge q with 4-velocity $\vec{u}$, in a field $\overleftrightarrow{F}$.

- In a specified frame, the field or Faraday tensor $\overleftrightarrow{F}$ has components

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & B^z & -B^y \\ -E^y/c & -B^z & 0 & B^x \\ -E^z/c & B^y & -B^x & 0 \end{bmatrix}$$

- Using $F^{\mu'\nu'} = \sum_{\alpha,\beta} \Lambda^{\mu'}{}_\alpha \Lambda^{\nu'}{}_\beta F^{\alpha\beta}$, we learned how the fields transform between frames:

$$\underset{\sim}{E}'_{\parallel} = \underset{\sim}{E}_{\parallel} \qquad \underset{\sim}{E}'_{\perp} = \gamma\left(\underset{\sim}{E}_{\perp} + \underset{\sim}{v} \times \underset{\sim}{B}_{\perp}\right)$$

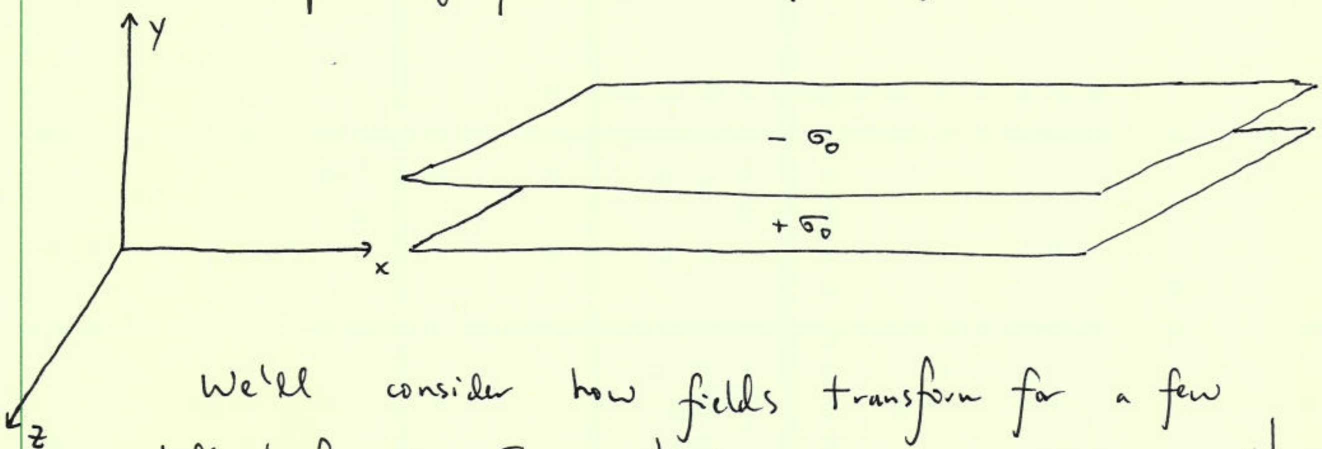
$$\underset{\sim}{B}'_{\parallel} = \underset{\sim}{B}_{\parallel} \qquad \underset{\sim}{B}'_{\perp} = \gamma\left(\underset{\sim}{B}_{\perp} - \underset{\sim}{v} \times \underset{\sim}{E}_{\perp}/c^2\right)$$

Primed frame moves with $\underset{\sim}{v}$ relative to unprimed.

This way of describing how the fields transform is the most general, and in most circumstances, the most direct way to figure out fields in "new" frame.

However, it is somewhat abstract. To get a more concrete picture of how things transform, let's examine some specific field arrangements.

Consider a pair of parallel sheets of charge:



We'll consider how fields transform for a few different frames. Even though we are considering a rather specific charge & field arrangement, our conclusions will be quite general since all that matters to understand how field changes is how field lines change.

Frame F_0: Sheets are at rest in this frame.

$$\boxed{\begin{aligned} \underset{\sim}{E}_0 &= \frac{\sigma_0}{\epsilon_0} \\ \underset{\sim}{B}_0 &= 0 \end{aligned}}$$

"Root" frame for this analysis.

Now consider a frame F which moves to the right with speed v_0 as seen by F_0. This has two effects:

1. The charge density is augmented by a factor $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$ due to length contraction. Let's define $\sigma = \gamma_0 \sigma_0$ = charge density in F.

2. Because they are in motion with $\underset{\sim}{v} = -v_0 \underset{\sim}{e}_x$ as seen in this frame, these sheets carry a surface current:

$$\underset{\sim}{K} = \sigma v_0 \underset{\sim}{e}_x \quad \text{(top)}$$
$$= -\sigma v_0 \underset{\sim}{e}_x \quad \text{(bottom)}$$

This generates a magnetic field:

$$\boxed{\begin{aligned} \underset{\sim}{E} &= \frac{\sigma}{\epsilon_0} \underset{\sim}{e}_y \\ \underset{\sim}{B} &= -\mu_0 \sigma v_0 \underset{\sim}{e}_z \end{aligned} \quad \text{in F}}$$

Check: If we use rules we derived with Faraday tensor, we see that this is compatible:

$$E^y = \gamma_0 E_0^y$$

$$B^z = -\gamma_0 v_0 E_0^y / c^2$$

Recalling that $1/c^2 = \mu_0 \epsilon_0$.

Now, just for fun, let's consider a frame F' that travels with $\underset{\sim}{V} = v \, \underset{\sim}{e}_x$ relative to F. What are the fields in this frame?

First, note that F' moves with

$$v' = \frac{v_0 + v}{1 + v v_0 / c^2}$$

relative to F_0. We can clearly deduce then that the fields in F' are given by

$$\underset{\sim}{E}' = \frac{\sigma'}{\epsilon_0} \underset{\sim}{e}_y$$

$$\underset{\sim}{B}' = -\mu_0 \sigma' v' \underset{\sim}{e}_z$$

where $\quad \sigma' = \gamma' \sigma_0 \, , \quad \gamma' = \dfrac{1}{\sqrt{1 - (v')^2 / c^2}}$

Fine. But, what's even more interesting is to figure out how the fields in F' are related to the fields in F: Gives us a way to see how electric & magnetic fields change between frames.

Compare equations:

$$E^{y'} = \left(\frac{\gamma'}{\gamma_0}\right) E^y = \left(\frac{\gamma'}{\gamma_0}\right) \frac{\sigma}{\epsilon_0}$$

$$\frac{\gamma'}{\gamma} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1-(v')^2/c^2}} = \frac{1 + vv_0/c^2}{\sqrt{1 - v^2/c^2}}$$

somewhat nasty algebra.

$$\equiv \gamma(1 + vv_0/c^2)$$

So, $E^{y'} = \gamma(1 + vv_0/c^2)\,\sigma/\epsilon_0$ $\qquad v_0\sigma = \frac{-B^z}{\mu_0}$

$$= \gamma\left(E^y - \frac{vB^z}{c^2\mu_0\epsilon_0}\right)$$

$$\rightarrow \boxed{E^{y'} = \gamma(E^y - vB^z)}$$

$$B^{z'} = \left(\frac{\gamma'}{\gamma_0}\right)\left(\frac{v'}{v_0}\right) B^z = -\left(\frac{\gamma'}{\gamma}\right)\mu_0\sigma v'$$

$$= -\gamma\,\cancel{(1 + vv_0/c^2)}\,\mu_0\sigma\frac{(v + v_0)}{\cancel{1 + vv_0/c^2}}$$

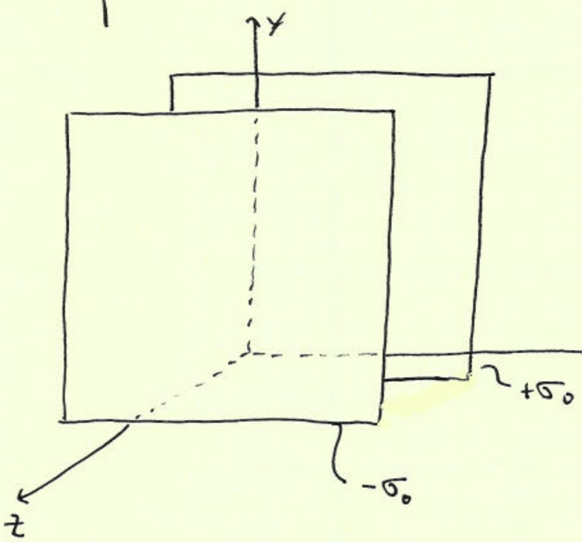
$$= -\gamma\mu_0 v_0\sigma - \gamma\mu_0\sigma v$$

$$= \gamma(B^z - \mu_0\epsilon_0 v E^y)$$

$$\rightarrow \boxed{B^{z'} = \gamma(B^z - vE^y/c^2)}$$

Perfect agreement! ... At least for those components. How do we do the others?

- To do E^z & B^y, just put capacitor in x-y plane rather than x-z plane:



In frame F_0, we now have $E_0^z = \frac{\sigma_0}{\epsilon_0}$;

in F we have

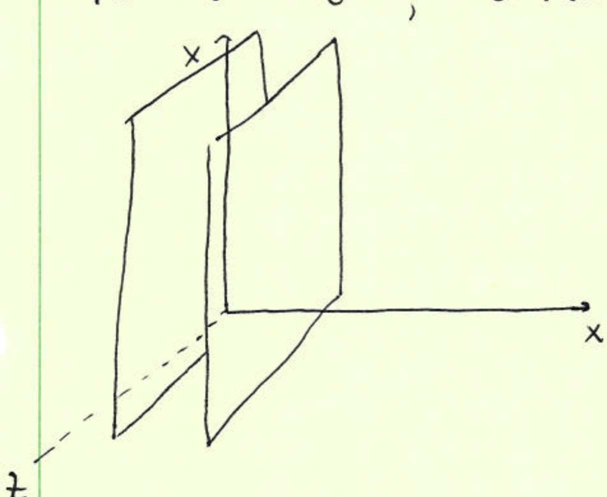
$$E^z = \frac{\sigma}{\epsilon_0}$$

$$B^y = \mu_0 \sigma v_0$$

Can see that the results are the same, provided we take $E^y \to E^z$, $B^z \to -E^y$:

$$\boxed{\begin{aligned} E^{z\prime} &= \gamma\,(E^z + vB^y) \\ B^{y\prime} &= \gamma\,(B^y + vE^z/c^2) \end{aligned}}$$

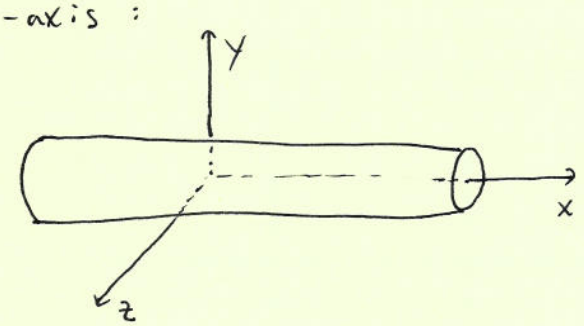
To do E^x, consider capacitor in y-z plane:



Motion along x makes plates closer... but has no effect on their charge density!

$$\boxed{E^{x\prime} = E^x}$$

For B^x, consider a solenoid oriented along the x-axis:



In "rest" frame, we have

$$B^x = \mu_0 n I$$

turns / length

In new frame,

$n \to n\gamma$

$\hookrightarrow$ length contraction boosts turns / length

but also $I \to I/\gamma$, thanks to time dilation.

$$\rightarrow \boxed{B^{x'} = B^x}$$

$\to$ 6 sets of equations that are identical to what we got using the Faraday tensor!

The abstract "field-oriented" way of transforming is exactly the same as a more concrete "source-oriented" way of doing it.

Final note: We've been doing this with the relative motion along x in all cases. We could do it for relative motion along y: Same formulas would hold provided we took

$$x \to y, \quad y \to z, \quad z \to x$$

Likewise if relative motion were along z, we promote

$$x \to z, \quad z \to y, \quad y \to x$$

A concise way to summarize the result of all of this:

$$\boxed{\begin{aligned} \underset{\sim}{E}'_{\parallel} &= \underset{\sim}{E}_{\parallel} \\ \underset{\sim}{E}'_{\perp} &= \gamma\left(\underset{\sim}{E}_{\perp} + \underset{\sim}{v} \times \underset{\sim}{B}_{\perp}\right) \\ \underset{\sim}{B}'_{\parallel} &= \underset{\sim}{B}_{\parallel} \\ \underset{\sim}{B}'_{\perp} &= \gamma\left(\underset{\sim}{B}_{\perp} - \underset{\sim}{v} \times \underset{\sim}{E}_{\perp}/c^2\right) \end{aligned}}$$

where $\underset{\sim}{E}_{\parallel}$ means "the component of $\underset{\sim}{E}$ parallel to $\underset{\sim}{v}$", $\underset{\sim}{E}_{\perp}$ means "the components of $\underset{\sim}{E}$ perpendicular to $\underset{\sim}{v}$", etc.

This "source-oriented" way of doing this calculation reminds us that we're not quite done putting E+M into covariant language: What about Maxwell's field equations,

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho / \epsilon_0 \qquad \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\partial \underset{\sim}{B} / \partial t \qquad \underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \mu_0 \epsilon_0 \, \partial \underset{\sim}{E} / \partial t$$

These equations are clearly not written in a covariant form! We've manifestly picked out time and space directions - want a more "spacetime" oriented way of handling this.

Source part is easy to do:

Let's define $J^1 = J^x$, $J^2 = J^y$, $J^3 = J^z$.

For the time like component, let's use charge density, with a factor of c to get the dimensions right:

$J^0 = c\rho$.

A really nice way to think of this current 4-vector is in analogy with the current density in regular 3-D E+M. In that case,

$$\underset{\sim}{J} = \rho \underset{\sim}{u}.$$

The new 4-vector we've defined looks just like this, but with ρ replaced by ρ_0, the charge density in the rest frame of the charge distribution; and with $\underset{\sim}{u}$ replaced by $\vec{u}$, the 4-velocity of the charges:

$$\vec{J} = \rho_0 \vec{u}$$

Notice that $\vec{J} \cdot \vec{J} = -\rho_0^2 c^2$ → a simple frame invariant way to see the charge density in the charge's rest frame!

Also automatically gives correct factors of γ when we transform between frames.

Next, just have to figure out how to get derivatives into $F^{\mu\nu}$ to make Maxwell work... ~~next lecture.~~

Now, onto the rest of Maxwell's equations: we clearly want to make an equation that is of the form

$$(\text{derivative})\,(\text{Fields}) = (\text{source})$$

The "Fields" are given by the Faraday tensor, $F^{\mu\nu}$. The "source" is the 4-current, J^μ (possibly with some constant at front to get the units right).

What do we use for the derivative? We need to figure out how to cast our derivative operators in index notation. Key question: how do derivatives behave under Lorentz transformations?

Fortunately, we already worked that out on pset #2! Two frames, F & F'; F' moves with $v\underset{\sim}{e}_x$ as seen by F. Showed on this pset that

$$\frac{1}{c}\frac{\partial}{\partial t'} = \frac{\gamma}{c}\frac{\partial}{\partial t} + \gamma\beta\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x'} = \gamma\beta\frac{1}{c}\frac{\partial}{\partial t} + \gamma\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

-or-

$$\frac{\partial}{\partial x^{0\prime}} = \gamma\frac{\partial}{\partial x^0} + \gamma\beta\frac{\partial}{\partial x^1}$$

$$\frac{\partial}{\partial x^{1\prime}} = \gamma\beta\frac{\partial}{\partial x^0} + \gamma\frac{\partial}{\partial x^1}$$

$$\frac{\partial}{\partial x^{2\prime}} = \frac{\partial}{\partial x^2}\,, \qquad \frac{\partial}{\partial x^{3\prime}} = \frac{\partial}{\partial x^3}$$

-or
$$\frac{\partial}{\partial x^{\mu'}} = \Lambda^{\alpha}{}_{\mu'} \frac{\partial}{\partial x^{\alpha}}$$

→ The derivative operator $\frac{\partial}{\partial x^{\mu}}$ transforms exactly like an object with indices in the "downstairs" position!

Shorthand often used: $\frac{\partial}{\partial x^{\alpha}} \equiv \partial_{\alpha}$

Then, $\partial_{\mu'} = \Lambda^{\alpha}{}_{\mu'} \partial_{\alpha}$.

Return now to our Maxwell equations:

$$(\text{derivative})\ F^{\mu\nu} = a\ J^{\mu}$$

some undetermined constant.

For "derivative", let's use ∂_{ν}, and sum over all ν:

$$\sum_{\nu=0}^{3} \partial_{\nu} F^{\mu\nu} = a J^{\mu}$$

Does this work? Let's take a look at a component to see: Plug in $\mu = 0$:

$$\frac{1}{c}\left[\frac{\partial(0)}{\partial t} + \frac{\partial E^{x}}{\partial x} + \frac{\partial E^{y}}{\partial y} + \frac{\partial E^{z}}{\partial z}\right] = a c \rho$$

or $$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = a c^2 \rho = \frac{a \rho}{\mu_0 \epsilon_0}$$

Choose $a = \mu_0$ ... The divergence of $\underset{\sim}{E}$ Maxwell equation!

Next, look at the $\mu = 1$ component:

$$\sum_{\nu=0}^{3} \partial_\nu F^{1\nu} = \mu_0 J^1$$

$$\rightarrow \frac{1}{c^2} \frac{\partial}{\partial t}\left(-E^x\right) + \frac{\partial B^z}{\partial y} + \frac{\partial}{\partial z}\left(-B^y\right) = \mu_0 J^x$$

$$\frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z} = \mu_0 J^x + \mu_0 \epsilon_0 \frac{\partial E^x}{\partial t}$$

$$\left(\underset{\sim}{\nabla} \times \underset{\sim}{B}\right)^x = \mu_0 J^x + \mu_0 \epsilon_0 \frac{\partial E^x}{\partial t}$$

Easy to show that this holds for the $\mu = 2$, $\mu = 3$ components:

$$\boxed{\begin{array}{l} \partial_\nu F^{\mu\nu} = \mu_0 J^\mu \quad \text{is} \\ \text{equivalent to} \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho / \epsilon_0 , \\ \qquad \underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \mu_0 \epsilon_0 \, \partial \underset{\sim}{E} / \partial t \end{array}}$$

How do we get the other two?

Kind of a trick involved here: There's not enough information in $F^{\mu\nu}$ to get the remaining "source free" Maxwell equations.

However, there is another way to fit the electric & magnetic fields into an antisymmetric tensor. Suppose we take $F^{\mu\nu}$, but swap the magnetic and electric field components:

$$F^{\mu\nu}\left(\underset{\sim}{E}/c \to \underset{\sim}{B},\ \underset{\sim}{B} \to -\underset{\sim}{E}/c\right)$$

$$= \begin{bmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z/c & E^y/c \\ -B^y & E^z/c & 0 & -E^x/c \\ -B^z & -E^y/c & E^x/c & 0 \end{bmatrix} \equiv G^{\mu\nu}$$

"Faraday dual tensor"

This tensor obeys all the same symmetries and transformations as the Faraday tensor, but does not give us a force law.

If we compute $\partial_\nu G^{\mu\nu} = 0$, we find

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0, \quad \frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{\nabla} \times \underset{\sim}{E} = 0$$

$\to$ the remaining Maxwell equations!

So: In covariant form,

$$\frac{dp^\mu}{d\tau} = q F^{\mu\nu} u_\nu$$

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

$$\partial_\nu G^{\mu\nu} = 0$$

completely summarizes the content and action of the electromagnetic field ... but does so in a way that makes very clear how quantities transform under Lorentz transformations.