Recap:

- Confirmed field transformation laws by considering how sources transform between frames
- Introduced 4-current: $\vec{J} = J^0 \vec{e}_t + J^1 \vec{e}_x + \cdots$ where $J^0 = c\rho$, $J^{x,y,z}$ are the usual current density
- Found that $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$ transforms just like any other quantity with index in the "downstairs" position.
- Using this, we formulated Maxwell's equations:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

$$\partial_\nu G^{\mu\nu} = 0$$

Interlude

We have now covered essentially all of special relativity! Now see how to formulate motion of bodies (action of forces, conservation of momentum and energy) and the most important fields that act on bodies in physics.

Key things worth emphasizing:

- Analysis is simplest when framed in terms of geometric objects in spacetime: 4-vectors, tensors, scalars. Reason it's simple: any indices get transformed by "hooking up" to a Lorentz transformation matrix. If there are no indices, there can be no transformation matrix: Your quantity must be a Lorentz invariant.

- Sometimes need to analyze things in a particular frame. Easy to do: Components of tensors and 4-vectors all tell us important physical quantities in the specified frame. Just need to pick them out.

Where we go from here:

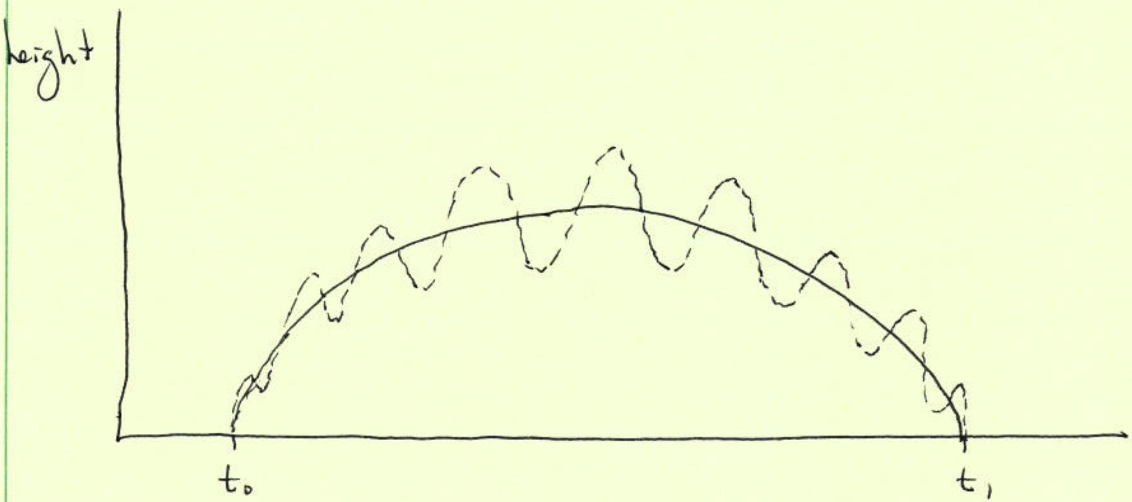
- Ultimate goal: An introduction to general relativity. The key insight here is that gravity can be understood as a consequence of geometry. In particular, the "flat" geometry of triangles, Pythagoras and the metric $\overleftrightarrow{\eta}$ must be replaced by a spacetime with curvature. Gravity will be intimately connected to the breakdown of the Pythagorean theorem.

- To take advantage of this introduction, we need to explore a tangent on "least action principles." This is a principle which allows us to reformulate classical mechanics in an elegant and powerful ... but perhaps rather abstract ... manner.

So, next few weeks: focus on least action principles, and set up why it is important in relativity.

Suppose I throw a ball in the air. At $t = t_0$, it is on the ground; at $t = t_1$, it is on the ground.

For $t_0 \le t \le t_1$, only gravity acts on it.



Thanks to 8.01, we know the trajectory is a parabola, not some crazy wiggly path. Simple, right?

It turns out there's another way to understand this path.

1. At any instant on the path, compute the ball's potential & kinetic energy.

2. Compute the <u>difference</u>, $L = K - U$; this is called "The Lagrangian."

3. Integrate over the interval:

$$S \equiv \int_{t_0}^{t_1} L \, dt \rightarrow \text{"The action"}$$

4. Repeat for <u>every</u> possible path that stretches between the end points.

Result: $\boxed{\text{The path described by Newton's laws corresponds to the minimum of } S \text{, over all possible trajectories: Nature chooses the path of LEAST ACTION!}}$

Cool! ... But why should we care? Seems a little complicated and abstract.

Three reasons why:

1. The minimization is not as crude as my caricature would make it seem! Powerful techniques from the "Calculus of variations" let us find the trajectory which extremizes the action quite simply.

2. Using these techniques, problems are actually often much <u>easier</u> to solve than using forces - particularly if a problem is subject to constraints in any way. Only need to compute a single scalar, rather a vector.

3. Least action principles appear to be intimately connected with how Nature is formulated. For example, one derives the principle of least action by examing how classical physics <u>emerges</u> from quantum physics: the action is intimately connected to the phase of a quantum wavefunction, and the trajectory which minimizes it is "most probable."

The Calculus of Variations

Basic idea: We have some function of a variable x and its derivative $\dot{x} = dx/dt$. We want to find extrema of

$$J = \int_{t_0}^{t_1} f\left[x(t), \dot{x}(t); t\right] dt$$

Our goal is to find $x(t)$ such that J is an extremum. We consider the function f and the endpoints, t_0 and t_1, to be fixed.

To proceed: imagine that there exists some $x_e(t)$ which gives us the extremum. Imagine that our current guess deviates from this "correct" choice as follows:

$$x(t) \equiv x(t; \alpha) = x_e(t) + \alpha A(t)$$

The function $A(t)$ is totally arbitrary, except that we require it to vanish at the endpoints: $A(t_0) = A(t_1) = 0$. The parameter α allows us to control how the variation $A(t)$ enters into our path $x(t;\alpha)$.

The correct path $x_e(t)$ is unknown - our goal is to figure out how to compute it, and to understand how f behaves when we are on this path.

Our basic idea is to ask how the integral J behaves when we are in the vicinity of the extremum. We know that ordinary functions are flat - have zero 1st derivative - when we are at an extremum.

Let us put $J(\alpha) = \int_{t_0}^{t_1} f\left[x(t;\alpha),\ \dot{x}(t;\alpha);t\right]\,dt$

We know that $\alpha = 0$ corresponds to the extremum by the definition of α. This isn't useful, though, since we don't know the $x(t)$ that this corresponds to.

However, we know $\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$ - perhaps this will be useful.

Let's try: $$\frac{\partial J}{\partial \alpha} = \int_{t_0}^{t_1}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \alpha}\right]dt$$

$$\frac{\partial x}{\partial \alpha} = A(t)\ ,\quad \frac{\partial \dot{x}}{\partial \alpha} = \frac{dA}{dt}$$

So, $$\frac{\partial J}{\partial \alpha} = \int_{t_0}^{t_1}\left[\frac{\partial f}{\partial x}A(t) + \frac{\partial f}{\partial \dot{x}}\frac{dA}{dt}\right]dt$$

Last term can be cleaned up using integration by parts:

$$\int_{t_0}^{t_1}\frac{\partial f}{\partial \dot{x}}\frac{dA}{dt}\,dt = \left.A(t)\frac{\partial f}{\partial \dot{x}}\right|_{t_0}^{t_1} - \int_{t_0}^{t_1}A(t)\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)dt$$

$A(t_0) = A(t_1) = 0$, so boundary term goes away.

So, $$\frac{\partial J}{\partial \alpha} = \int_{t_0}^{t_1} A(t) \left[\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \right] dt$$

$= 0$ if we are at an extremum of J.

The function $A(t)$ is totally arbitrary except at $t = t_0$ and $t = t_1$, where it vanishes. We require $\partial J / \partial \alpha$ to vanish for all $A(t)$, which means the quantity in brackets must be zero:

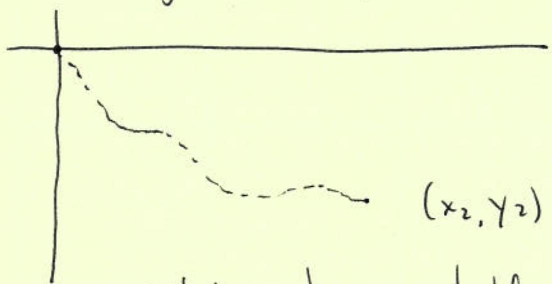
$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} = 0}$$

This known as Euler's equation - 1st derived by Euler in 1744.

Properly applied, it tells us how we find the trajectory which extremizes the integral J.

Classic example: The brachistochrone ("shortest time")

A bead starts at $x_1 = 0$, $y_1 = 0$ and slides down a wire frictionlessly, reaching x_2, y_2.



What shape should the wire have in order to reach (x_2, y_2) in as little time as possible?

Use Euler to minimize travel time:

$$T = \int_1^2 \frac{ds}{v}$$

$$ds = \sqrt{dx^2 + dy^2}$$
$$= dy\sqrt{1 + (x')^2} \qquad x' = \frac{dx}{dy}$$

$$v = \sqrt{2gy}$$ (assuming bead starts at rest)

$$\rightarrow \quad T = \int_0^{y_2} \sqrt{\frac{1 + (x')^2}{2gy}}\, dy$$

Next, apply Euler. We use

$$f = \sqrt{\frac{1 + (x')^2}{2gy}}$$

and change $t \rightarrow y$, $\dot{x} \rightarrow x'$

Euler becomes $\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x'} = \frac{1}{\sqrt{2gy}} \frac{x'}{\sqrt{1 + (x')^2}}$$

So, $\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$ becomes

$$\frac{d}{dy}\left[\frac{1}{\sqrt{2gy}} \frac{x'}{\sqrt{1 + (x')^2}}\right] = 0$$

$$\rightarrow \quad \frac{1}{\sqrt{2gy}} \frac{x'}{\sqrt{1 + (x')^2}} = \text{Constant}$$

Square both sides; make a special choice for the constant:

$$\frac{(x')^2}{2gy\,(1 + (x')^2)} = \frac{1}{4gA}$$

$$\rightarrow \quad \left(\frac{dx}{dy}\right)^2 = \frac{y/2A}{1 - y/2A}$$

$$= \frac{y^2}{2Ay - y^2}$$

$$\rightarrow \quad x = \int_0^{y_2} \frac{y\,dy}{\sqrt{2Ay - y^2}}$$

To solve this, change variables:

$$y = A(1-\cos\theta), \qquad dy = A\sin\theta\, d\theta$$

$$\begin{aligned} 2Ay - y^2 &= 2A^2 - 2A^2\cos\theta - A^2 + 2A^2\cos^2\theta - A^2\cos^2\theta \\ &= A^2(1-\cos^2\theta) \\ &= A^2\sin^2\theta \end{aligned}$$

$$\rightarrow \quad x = \int_0^\theta A(1-\cos\theta)\, d\theta = A(\theta - \sin\theta)$$

Full solution: The brachistochrone is described by

$$\boxed{\begin{aligned} x &= A(\theta - \sin\theta) \\ y &= A(1-\cos\theta) \end{aligned}} \qquad \theta \in [0, \theta_{max}]$$

"Cycloid"

Not quite done: Given x_2 and y_2, need to solve for A and θ_{max}. But this is not too difficult.