Recap:

- Discussed how to compute the "extremum" of an integral:

If

$$J = \int_{t_0}^{t_1} f\left[x(t), \dot{x}(t); t\right] dt$$

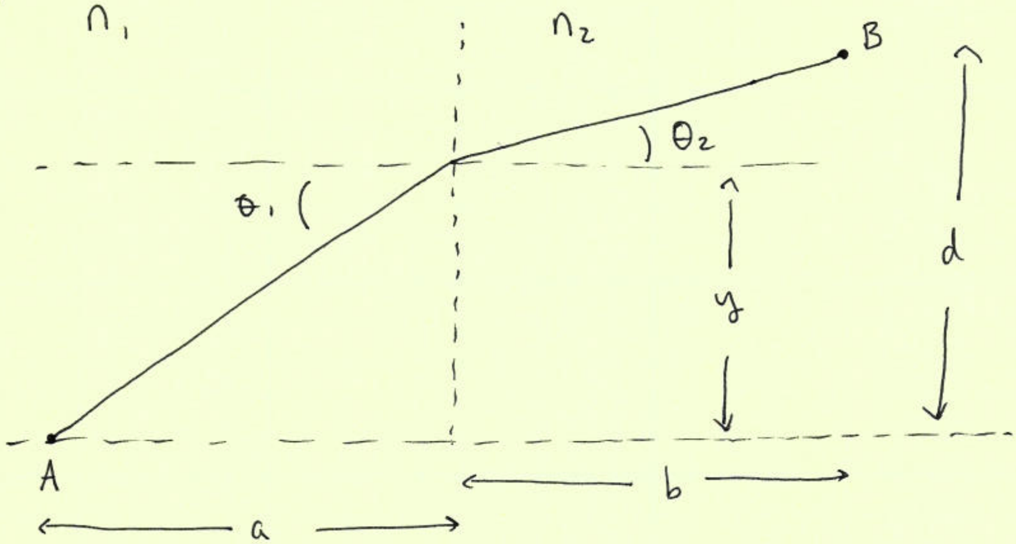
then the "trajectory" $x(t)$ can be found by requiring that f satisfy <u>Euler's</u> <u>equation</u>:

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$$

- Relevance to us: Physical systems are often characterized by some integrated quantity for which Nature "chooses" the path that minimizes it. These techniques allow us to figure out a path to the solution that is relatively quick and painless.

Famous example of this: Fermat's principle and light.

Light incident on some medium will "chose" the path which makes the travel time shortest.



Light travels at $v_1 = c/n_1$ in medium 1, and at $v_2 = c/n_2$ in medium 2.

Total travel time: $$T = \frac{\sqrt{a^2+y^2}}{v_1} + \frac{\sqrt{b^2+(d-y)^2}}{v_2}$$

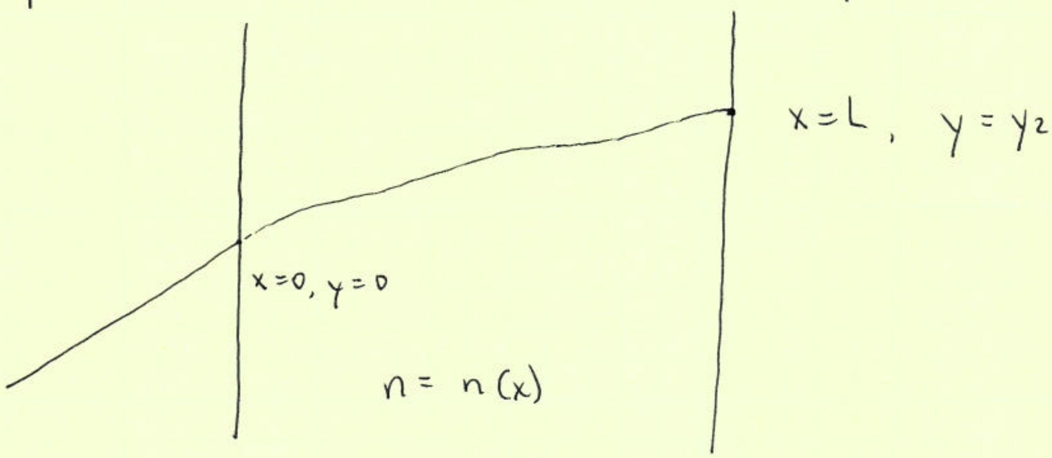
Minimize: $$\frac{dT}{dy} = \frac{y}{v_1\sqrt{a^2+y^2}} - \frac{(d-y)}{v_2\sqrt{b^2+(d-y)^2}}$$

$$= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$$

$$\rightarrow \boxed{\frac{v_1}{v_2} = \frac{n_2}{n_1} = \frac{\sin\theta_1}{\sin\theta_2}}$$

Snell's law.

More complicated version: If index of refraction varies with position, use Euler equation to analyze:



$$T = \int_0^L dx \; \frac{\sqrt{1+(y')^2}}{c/n(x)}$$

Apply Euler: $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\dfrac{\partial f}{\partial y'} = 0$, $f = \dfrac{n(x)\sqrt{1+(y')^2}}{c}$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{y'\,n(x)}{c\sqrt{1+(y')^2}}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \rightarrow \frac{y'\,n(x)}{c\sqrt{1+(y')^2}} = \text{Constant}$$

$$\rightarrow \quad \frac{dy}{dx} = \frac{A}{\sqrt{n(x)^2 - A^2}}$$

Match slope at $x = 0$ (choose A appropriately), then integrate differential equation to build trajectory light follows.

Key application in physics: Lagrangian mechanics.

All of mechanics can be reproduced by looking for minima of the "action" S, defined by

$$S = \int L \, dt$$

where $L = K - U$ = (kinetic energy) - (potential energy) is the "Lagrangian."

For detailed justification of this, take 8.09! To get some intuition, consider a 1-D problem:

$$K = \frac{1}{2} m \dot{x}^2 \, , \quad U = U(x)$$

Apply Euler: $\displaystyle \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

$\displaystyle \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} \equiv F$, force due to gradient of potential energy.

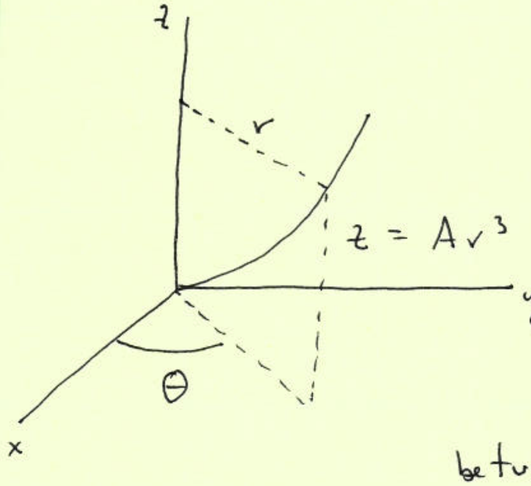
$\displaystyle \frac{\partial L}{\partial \dot{x}} = m\dot{x} \equiv p$, momentum of a mass m with speed $\dot{x}$.

→ Euler equation implies $\boxed{F = \dfrac{dp}{dt}}$ Usual Newtonian force law!

Beyond this example, Lagrangian can depend on any number of coordinates.

Particular value: Allows us to easily introduce <u>constraints</u>.

Example: A bead slides without friction along a wire bent into the shape $z = A r^3$



When the wire is spun about the z axis, centrifugal force moves the ~~bead~~ bead out to a distance $r = R$ from the axis. What is the relationship between the radius R and the angular velocity ω?

Assume initially that the bead is free to move however it "wants". Then,

$$K = \frac{1}{2} m \left(\dot{r}^2 + \dot{z}^2 + r^2 \dot{\theta}^2 \right)$$

Now, insert constraints: $\dot{\theta} = \omega$, $\dot{z} = \dot{(A r^3)} = 3 A r^2 \dot{r}$

So, $$K = \frac{1}{2} m \left(\dot{r}^2 + 9 A^2 r^4 \dot{r}^2 + r^2 \omega^2 \right)$$

The potential energy is $U = mgz = mg A r^3$

So,
$$L = K - U$$
$$= \frac{1}{2} m \left(\dot{r}^2 + 9 A^2 r^4 \dot{r}^2 + r^2 \omega^2 \right) - mg A r^3$$

→ Notice it only depends on r, $\dot{r}$!

Apply Euler's equations:

$$\frac{\partial L}{\partial r} = \frac{1}{2}m\left(36A^2r^3\dot{r}^2 + 2r\omega^2\right) - 3mgAr^2$$

$$= mr\left(18A^2r^2\dot{r}^2 + \omega^2 - 3gAr\right)$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m\left(2\dot{r} + 18A^2r^4\dot{r}\right)$$

$$= m\left(\dot{r} + 9A^2r^4\dot{r}\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\left(\ddot{r} + 9A^2r^4\ddot{r} + 36A^2r^3\dot{r}^2\right)$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$\rightarrow -\ddot{r}\left(1 + 9A^2r^4\right) + \dot{r}^2\left(18A^2r^3 - 36A^2r^3\right)$$

$$+ r\left(\omega^2 - 3gAr\right) = 0$$

-or-

$$\ddot{r}\left(1 + 9A^2r^4\right) + \dot{r}^2\left(18A^2r^3\right)$$

$$+ r\left(3gAr - \omega^2\right) = 0$$

When in equilibrium, $\dot{r} = 0$, $\ddot{r} = 0$, $r = R$:

$$3gAR - \omega^2 = 0$$

$$\rightarrow R = \frac{\omega^2}{3gA}$$

-or-

$$\omega = \sqrt{3gAR}$$

Particularly important feature for our purposes: The Lagrangian helps us identify quantities which are conserved in our system.

To proceed, useful to make a definition. We saw earlier in 1-D that $\partial L / \partial \dot{x} = m\dot{x} = p$, the momentum. This is so useful that we define a "generalized momentum" this way for any coordinate system:

$$p_q \equiv \frac{\partial L}{\partial \dot{q}}$$

$q \equiv$ some coordinate in our analysis.

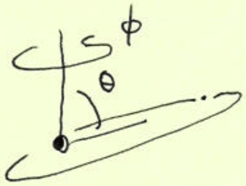
Note that p_q might not even have the same units as momentum - but, it will play a role like momentum in our analysis. We call it the momentum "conjugate" to the coordinate q.

Now, suppose the Lagrangian is independent of coordinate q: $\partial L / \partial q = 0$. Then, by Euler's equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \rightarrow \boxed{p_q \equiv \frac{\partial L}{\partial \dot{q}} \text{ is constant}}$$

In words: if L is independent of coordinate q, the momentum conjugate to q is a constant.

Example: Consider a body in orbit in a spherical potential:



$$K = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right)$$

$$U = U(r)$$

$$L = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) - U(r)$$

$$\frac{\partial L}{\partial r} \neq 0 \qquad \frac{\partial L}{\partial \theta} \neq 0 \qquad \frac{\partial L}{\partial \phi} = 0$$

$$\rightarrow \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi}$$

$$= \text{constant}$$

Symmetry of Lagrangian lets us immediately identify p_ϕ as a conserved constant in this problem. p_ϕ is in fact the angular momentum associated with this motion.

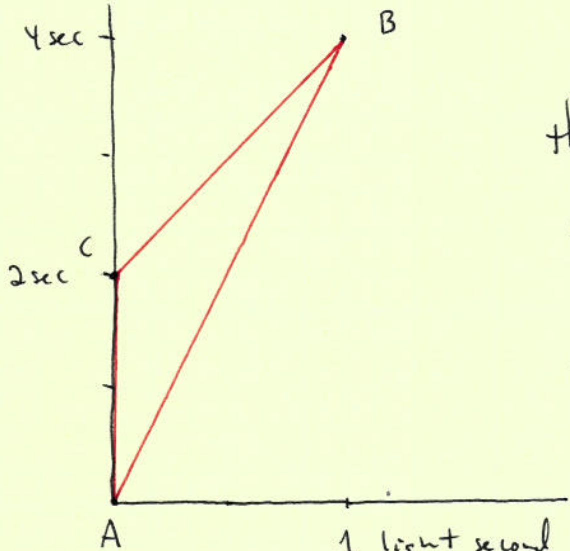
Result is an example of "Noether's theorem": Any symmetry of the system is associated with a conservation law.

Key thing that we will shortly take advantage of!

Application relevant to this class: Computing trajectories of bodies in spacetime.

Imagine some body travels from event A ($x = 0$, $t = 0$) to event B ($x =$ 1 light second, $t = 4$ sec)

Consider two paths that it uses to do this: Direct path from $A \to B$, and an indirect path through event C ($x = 0$, $t = 2$ sec):



Question: On which path does the body age more $A \to B$, or $A \to C \to B$?

Answer this by comparing proper time accumulated on the two trajectories: Easily done using the fact that

$$\Delta s^2 = -c^2 \Delta \tau^2.$$

$A \to B$: $\Delta \tau = \left[(4\,\text{sec})^2 - (1\,\text{light sec}/c)^2 \right]^{1/2}$

$= \sqrt{15}\ \text{sec} \simeq 3.87\ \text{sec}$

$A \to C$: $\Delta \tau = 2\,\text{sec}$

$C \to B$: $\Delta \tau = \left[(2\,\text{sec})^2 - (1\,\text{light sec}/c)^2 \right]^{1/2}$

$= \sqrt{3}\ \text{sec} \simeq 1.73\ \text{sec}.$

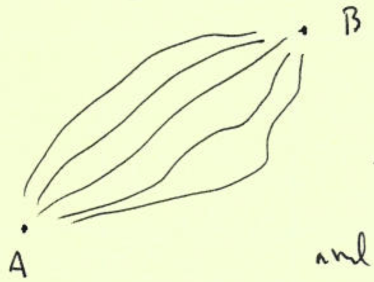
So $\Delta\tau_{A\to B} = 3.87$ sec is greater than

$\Delta\tau_{A\to C\to B} = 3.73$ sec.

Is there a lower bound on the aging we can experience? Yes: $\Delta\tau = 0$! This is the limit we approach if the trajectory is made up of a sequence of segments that go at the speed of light.

→ We can never actually achieve this limit. As a consequence, when we consider proper time, the interesting extremum is the <u>maximum</u>.

Using calculus of variations, we can determine which path gives us the maximum of accumulated proper time:



Which has largest $\Delta\tau$?

Take our formula for invariant interval, and consider the infinitesimal limit:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$\to \quad d\tau = dt\sqrt{1 - \frac{(dx/dt)^2}{c^2}}$$

$$\tau_{A\to B} = \int_A^B dt\sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

Put $f(x, \dot{x}) = \sqrt{1 - \frac{\dot{x}^2}{c^2}}$

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial \dot{x}} = \frac{-\dot{x}/c^2}{\sqrt{1 - \dot{x}^2/c^2}}$$

$$\rightarrow \quad \frac{d}{dt}\left[\frac{-\dot{x}/c^2}{\sqrt{1-\dot{x}^2/c^2}}\right] = 0$$

$$\rightarrow \quad \boxed{\dot{x} = \text{constant}}$$

Include y, and z, find 3 separate equations which tell us in addition

$$\boxed{\begin{aligned} \dot{y} &= \text{constant} \\ \dot{z} &= \text{constant} \end{aligned}}$$

In other words, "maximal aging" means moving at constant speed - no forces or accelerations.

→ The unaccelerated trajectory is the one that maximizes the observer's accumulated proper time.