Recap:

- Continued discussion of "least" principles in physics:

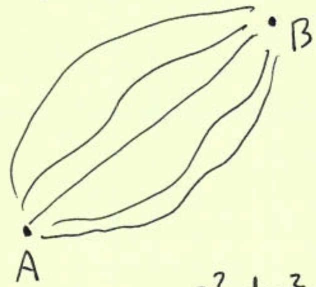
  - Fermat's principle + "least time" → Snell's law for light.

  - Lagrangian mechanics: $L = K - U$ plus
    $$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$
    reproduces Newton's laws for q a generalized coordinates describing motion.

  - Very important consequence: If $\partial L/\partial q = 0$, then the momentum conjugate to coordinate q, $p_q \equiv \frac{\partial L}{\partial \dot{q}}$, is conserved.

  - Principal of maximal aging: Given two events in spacetime, A + B, the unaccelerated trajectory between them represents the trajectory along which the most proper time accumulates.

Euler's equation is condition for extremum: Apply to proper time, find trajectories that are straight lines in Cartesian coordinates.



To find path which accumulates largest $\Delta\tau$, write

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$\rightarrow \quad d\tau = dt\sqrt{1 - \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{c^2}}$$

$$\tau_{A\to B} = \int_A^B dt\sqrt{1 - \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{c^2}}$$

$$= \int_A^B dt \; f(x, y, z; \dot{x}, \dot{y}, \dot{z})$$

Apply Euler: Since $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$,

we get $\frac{\partial f}{\partial \dot{x}} = \frac{-\dot{x}/c^2}{\sqrt{1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2}} = \text{constant}$

$$\frac{\partial f}{\partial \dot{y}} = \frac{-\dot{y}/c^2}{f} = \text{constant}$$

$$\frac{\partial f}{\partial \dot{z}} = \frac{-\dot{z}/c^2}{f} = \text{constant}$$

$$\rightarrow \quad \dot{x} = \text{constant}$$

$$\dot{y} = \text{constant}$$

$$\dot{z} = \text{constant}$$

Hence, motion in a straight line.

Trajectories of extremal aging are called "geodesics," in analogy with trajectories of extremal length in space, like great circles on a sphere.

Geodesics look like straight lines if we use "straight coordinates" - ie, Cartesian or inertial coordinates. What if we use different coordinates? For instance, if we work in spherical coordinates, we have

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

The same as our "usual" form, provided we have

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

We then have $d\tau = \int dt \; f(r, \theta, \phi; \dot{r}, \dot{\theta}, \dot{\phi})$

where

$$f = \sqrt{1 - \frac{[(\dot{r})^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2]}{c^2}}$$

Euler equation $\frac{\partial f}{\partial q} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}} = 0$ will

give equations describing motion in r, θ, ϕ ... but it will be <u>much</u> more complicated. Notice that $\frac{\partial f}{\partial r} \neq 0$, $\frac{\partial f}{\partial \theta} \neq 0$ ~ p_r, p_θ are <u>not</u> constant.

Defer additional discussion for several weeks - this procedure is key to describing motion in general relativity.

Incidentally, behavior of geodesics allows us to understand why I sometimes refer to the spacetime of special relativity as "flat." Initially, we set this up by analogy:

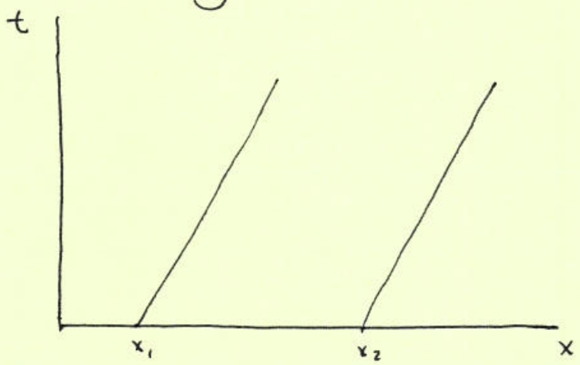
On a 2-D flat surface, $\Delta s^2 = \Delta x^2 + \Delta y^2$

In 3-D, $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$

→ same idea, continued into new direction.

In spacetime, $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ likewise continues the idea.

More rigorous way to see this: Examine two geodesics.



Two trajectories that start out parallel to one another will clearly never cross: they remain parallel forevermore.

This defines flat. Intimately tied to the fact that this is a geometry for which Euclid's 5^{TH} postulate - the "parallelism postulate" - is true.

A curved trajectory will then be one for which this is not true: Initially parallel trajectories will diverge or cross, like lines of longitude on a globe.

Advanced topic: Field Theories

Goal of this discussion: Develop a basic understanding of how we build theories of the fields that characterize the most important interactions we know of in physics.

Field: Quantity that varies through space and time, usually interacts in some way to carry a force.

Example: Electricity and magnetism. A lot of our previous effort was spent working out how to write the laws of E+M in such a way that they manifestly respect the laws of special relativity.

We did this in a somewhat post facto way: we already knew Maxwell's equations, we picked things out in a way that reproduce them. Fine ... if you already know the answer. How do you proceed if you do not know the answer?

Our approach: Develop a Lagrangian that describes how fields behave. For particles, we have

$$L(x,y,z;\dot{x},\dot{y},\dot{z}) = K - U$$

$$\rightarrow \quad \frac{\partial L}{\partial(x,y,z)} - \frac{d}{dt}\frac{\partial L}{\partial(\dot{x},\dot{y},\dot{z})} = 0$$

We would like to define a similar quantity for fields,

$$L = L(\underbrace{\Phi, \partial\Phi})$$

Means schematically "Function of fields and derivative of fields"

Requiring that the associated action be an extremum should yield equations which describe how to find fields which correspond to "good physics."

Principle we'll follow: Since fields fill all of space, we actually compute the Lagrange <u>density</u> rather than the Lagrangian:

$$L = \int dV \, \mathcal{L}$$

This means the action is given by integrating $\mathcal{L}$ over all of spacetime:

$$S = \int L \, dt = \int dt \, dV \, \mathcal{L}$$

The action must be the same in all frames - otherwise, we could conceivably pick at a preferred frame.

Under Lorentz transformation, $dV \to dV/\gamma$ due to length contraction; but $dt \to \gamma \, dt$. Hence "spacetime" 4-D volume element is invariant.

Since S is also invariant, we conclude that

$\mathcal{L}$ must be a Lorentz invariant quantity.

→ Great! Since $\mathcal{L}$ is a simple scalar, and respects Lorentz invariance, it will give us a simple but powerful tool for organizing information about the field.

More generally, we build $\mathcal{L}$ in such a way that it builds in the symmetries that describe physical laws.

- Lorentz symmetry ... for now
- Principle of equivalence (gravity)
- Supersymmetry ?
- Large extra dimensions ?
- ...

For now, we confine ourselves to Lorentz symmetry, which tells us that the result cannot pick out some frame as "special."

Working in index notation, this means that we cannot have any "leftover" free indices: everything must be contracted and summed over.

Let's imagine that our interaction is described by a single field Φ. How do we write out its Lagrange density?

$$\mathcal{L} = \mathcal{L}\left[\Phi, \text{“}\partial\Phi\text{”}\right]$$

simple enough   but what's that?

With mechanics, we used time derivatives here... but in relativity, we shouldn't pick out "time" as special, it will mix with "space" under Lorentz transformations.

So, let's consider all possible derivatives:

$$\mathcal{L} = \mathcal{L}\left[\Phi, \frac{\partial\Phi}{\partial x^\mu}\right] = \mathcal{L}\left[\Phi, \partial_\mu\Phi\right]$$

Now, how do we find the Φ such that $S = \int dt\, dV\, \mathcal{L}$ is an extremum?

Exactly as we did it before! We imagine that there is some Φ that extremizes S, and we see how things behave as things deviate slightly from that:

$$\Phi \rightarrow \Phi(\alpha) = \Phi_e + \alpha A$$

↳ Now a field too.

$$S(\alpha) = \int dt\, dV\, \mathcal{L}\left[\Phi(\alpha), \partial_\mu\Phi(\alpha)\right]$$

On extremum, we have $\frac{\partial S}{\partial \alpha} = 0$:

$$0 = \int dt\, dV \left[\frac{\partial \mathcal{L}}{\partial \Phi} \frac{\partial \Phi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \frac{\partial (\partial_\mu \Phi)}{\partial \alpha} \right]$$

Last term is a bit weird:

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \frac{\partial (\partial_\mu \Phi)}{\partial \alpha} = \frac{\partial \mathcal{L}}{\partial (\partial \Phi / \partial x^0)} \frac{\partial}{\partial \alpha} \left(\frac{\partial \Phi}{\partial x^0} \right) + \dots + \frac{\partial \mathcal{L}}{\partial (\partial \Phi / \partial x^3)} \frac{\partial}{\partial \alpha} \left(\frac{\partial \Phi}{\partial x^3} \right)$$

(Using Einstein summation convention here for conciseness.)

$$\frac{\partial (\partial_\mu \Phi)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left(\frac{\partial \Phi}{\partial x^\mu} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial \Phi_e}{\partial x^\mu} + \alpha \frac{\partial A}{\partial x^\mu} \right)$$

$$= \frac{\partial A}{\partial x^\mu} \equiv \partial_\mu A$$

$$\rightarrow \quad 0 = \int dt\, dV \left[\frac{\partial \mathcal{L}}{\partial \Phi} A + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \partial_\mu A \right]$$

Next: Massaging this a bit to extract a condition on $\mathcal{L}$.