Recap:

- Additional discussion of <u>geodesics</u>: Trajectories in spacetime that extremize "aging," ie the proper time accumulated on the trajectory.

  In special relativity, trajectories that start parallel will remain parallel forever. Means that the geometry of spacetime is "flat."

  Curvature means that trajectories which start parallel are focused and cross, or are dispersed.

- Began discussion of field <u>theories</u>: Will use Lagrangian density as tool to organize all information about field into Lorentz invariant form.

Initial example: Scalar field. We put

$$\mathcal{L} = \mathcal{L}[\Phi, \partial_\mu \Phi]$$

We assume some Φ_e gives an extremum of the action, $S = \int dt\, dV\, \mathcal{L}$, and examine deviations about this:

$$\Phi(\alpha) = \Phi_e + \alpha A$$

$$S(\alpha) = \int dt\, dV \Big[\mathcal{L}[\Phi(\alpha), \partial_\mu \Phi(\alpha)] \Big]$$

$$\frac{\partial S}{\partial \alpha} = 0 \quad \longrightarrow \quad 0 = \int dt\, dV \left[\frac{\partial \mathcal{L}}{\partial \Phi} A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu A \right]$$

(Note: Assuming Einstein summation convention.)

The 2ND term here can be massaged a bit:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu A = \partial_\mu \left[A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right] - A\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right)$$

With this, $\partial S / \partial \alpha = 0$ means

$$0 = \int dt\, dV \left[A \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \right) + \partial_\mu \left[A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right] \right]$$

Now, focus on the final term here:

$$\partial_\mu \left[A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right] = \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \left[A \frac{\partial \mathcal{L}}{\partial(\partial \Phi / \partial x^\mu)} \right]$$

With a little effort, one can prove that

$$\frac{\partial \mathcal{L}}{\partial(\partial \Phi / \partial x^\mu)}$$

represents the components of a 4-vector. (Proof: transform to some new frame. You'll see that the components in the new frame are related to these components with the usual Lorentz matrix for 4-vector components.) Given this, let's write

$$A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \equiv \Delta^\mu$$

This final term is thus of the form

$$\int dt\, dV \;\; \partial_\mu \Delta^\mu \qquad .$$

This should remind you of the divergence theorem you probably 1st saw in 8.02 / 8.022:

$$\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV = \int_S \underset{\sim}{F} \cdot d\underset{\sim}{a}$$

V = some volume S = The surface of that volume's boundary.

A similar theorem can be proven for a 4-dimensional region of spacetime:

$$\int_{V_4} dt\, dV\ \partial_\mu F^\mu = \int_{S_3} F^\mu\, d\Sigma_\mu$$

V_4 = 4-D "volume" of spacetime
S_3 = 3-D "surface" of that volume

$d\Sigma_\mu$ = 3-D "surface element" of the boundary of a spacetime volume.

The right-hand side of this equation represents a flux of the field F^μ through the surface S_3.

Return to our action integral:

$$0 = \int dt\, dV \left[\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \right) \right]$$

$$+ \int dt\, dV\ \partial_\mu \Delta^\mu$$

What is the 4-D volume? ALL OF SPACETIME!

What's the boundary of this? IT HAS NO BOUNDARY!!

Hence, $\int dt\, dV\, \partial_\mu \Delta^\mu = 0$.

Since A is an arbitrary function, we must have

$$\boxed{\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) = 0 \qquad \text{-or} \\ &\frac{\partial \mathcal{L}}{\partial \Phi} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial \Phi / \partial x^\mu)} \right) = 0 \end{aligned}}$$

Worth comparing this to the particle Lagrangian:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$\mathcal{L}$ vs L : Density since fields fill all of space.

Φ vs x : Field gives extremum, rather than trajectory.

$\frac{\partial}{\partial x^\mu}$ vs $\frac{d}{dt}$: We cannot pick out one direction in spacetime as "special."

Besides these details, results are pretty much the same!

More details: How do we actually make the Lagrangian density? In analogy to the particle Lagrangian, we'd like to say

$$\mathcal{L} = \mathcal{K} - \mathcal{U}$$

$\mathcal{U}$ → potential energy density.

$\mathcal{K}$ → kinetic energy density

We might guess $\mathcal{K} = \frac{1}{2}\left(\frac{\partial \Phi}{\partial t}\right)^2$ in analogy with $K \propto (\dot{x})^2$ for a particle.

However, this is not Lorentz invariant. To make it so, we also include "gradient energy" - energy associated with the spatial variation of Φ. To get a Lorentz invariant result, we put

$$\mathcal{K} = \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial \Phi}{\partial t}\right)^2 - \left(\frac{\partial \Phi}{\partial x}\right)^2 - \left(\frac{\partial \Phi}{\partial y}\right)^2 - \left(\frac{\partial \Phi}{\partial z}\right)^2\right]$$

$$= -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha \Phi\, \partial_\beta \Phi$$

Potential energy? Defer to later ... for now, we'll just require it to be some function of Φ alone.

Then,
$$\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha \Phi\, \partial_\beta \Phi - \mathcal{U}(\Phi)$$

Now, require that this satisfy Euler equation:

$$\frac{\partial \mathcal{L}}{\partial \Phi} = - \frac{\partial \mathcal{U}}{\partial \Phi}$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} = -\frac{1}{2} \eta^{\alpha\beta} \frac{\partial}{\partial(\partial_\mu \Phi)} \left[\partial_\alpha \Phi \; \partial_\beta \Phi \right]$$

How the heck do we handle this? Here's a simple rule: If you differentiate an object with an index, V_α, using the same object but perhaps a different index:

$$\frac{\partial V_\alpha}{\partial V_\mu} = \delta^\mu{}_\alpha = 1 \quad \text{if } \mu = \alpha$$
$$= 0 \quad \text{if } \mu \neq \alpha$$

For intuition, imagine $V_\alpha = X_\alpha$. Each coordinate is independent of all others, so the partial derivative is either zero or one.

With this in mind,

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} = -\frac{1}{2} \eta^{\alpha\beta} \left[\delta^\mu{}_\alpha \, \partial_\beta \Phi + \delta^\mu{}_\beta \, \partial_\alpha \Phi \right]$$

$$= -\frac{1}{2} \left[\eta^{\mu\beta} \partial_\beta \Phi + \eta^{\alpha\mu} \partial_\alpha \Phi \right]$$

$$= - \partial^\mu \Phi .$$

So: $\displaystyle \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \right) = 0$

$$\rightarrow - \mathcal{U}(\Phi) + \partial_\mu \partial^\mu \Phi = 0$$

$$\partial_\mu \partial^\mu \Phi = \Box \Phi$$

↳ The "wave equation" operator

$$\rightarrow \quad \boxed{\Box \Phi - \frac{\partial U}{\partial \Phi} = 0}$$

Now, how about the potential? Depends on some details of your system.

Very simple case: $U = 0 \rightarrow \boxed{\Box \Phi = 0}$

"Klein-Gordon Equation"

Less simple case: Imagine a potential that looks like

$$U \propto \Phi^2$$

From the fact that Φ is analogous to position for a particle, we can think of this as similar to a harmonic oscillator:

$$U(x) = \frac{1}{2} k x^2$$
$$= \frac{1}{2} m \omega_0^2 x^2$$

Notice that the proportionality constant is related to the mass on the spring.

Inspired by this, let's see what happens when we pick a similar form here:

$$\mathcal{U} = \frac{1}{2} \beta^2 m^2 \Phi^2$$

Why m^2? And what is β?

I choose this form because I know $\beta^2 m^2$ ~~must~~ must be $1/(\text{length})^2$ to be consistent with the units of the kinetic term. It turns out that quantum mechanics gives us a very natural way to get 1/length from mass:

$$\lambda_c = \frac{h}{mc} \quad : \quad \text{Compton wavelength associated with mass } m.$$

$$\text{or } k_c = \frac{mc}{\hbar} = \frac{2\pi}{\lambda_c}$$

h = Planck's constant $= 6.626 \times 10^{-34}$ J-s

$\hbar = h/2\pi$

So, let's try $\mathcal{U} = \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \Phi^2$

then, Euler equation yields

$$\boxed{\Box \Phi - \frac{m^2 c^2}{\hbar^2} \Phi = 0}$$

"Massive Klein-Gordon Equation"

What does this mean?

To get some idea, look at this for $m=0$, static field:

$$\Box \Phi - \frac{m^2c^2}{\hbar^2}\Phi = 0 \quad \rightarrow \quad \nabla^2 \Phi = 0$$

Solution: $$\Phi = \frac{Q}{r}$$

where Q is some kind of charge associated with the field, r is distance.

Key point: Falls off as $1/r$, which means there is finite flux through arbitrarily large spheres.

Now, consider $m \neq 0$: For static field,

$$\nabla^2 \Phi - \frac{m^2c^2}{\hbar^2}\Phi = 0$$

$$\Phi = \frac{Q}{r}\exp\left[-mcr/\hbar\right]$$

→ Much sharper falloff! This makes it a "short-range" interaction.

Two key lessons:

1. Whenever the Lagrangian contains terms like $(\text{field})^2$, the coefficient tells us about the mass associated with the field.

2. The more massive the field, the shorter the field's range!

Massive fields play a huge role in particle physics setting up interactions that are confined to a small region.

Another example: Electricity and magnetism.

How do we make a Lagrangian for this?
We already know what energy densities associated with $\underset{\sim}{E}$ + $\underset{\sim}{B}$ fields. $\mathcal{U}_E = \frac{\epsilon_0}{2} E^2$, $\mathcal{U}_B = \frac{B^2}{2\mu_0}$

Let us choose the 4-components of the vector potential as the "field" that we will vary ~~###~~ in our Lagrangian. Now, treat "electric" energy as kinetic term, "magnetic" as potential term, and we get

$$\mathcal{L}_{EM} = \mathcal{U}_E - \mathcal{U}_B$$
$$= \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} - B^2\right)$$
$$= -\frac{1}{4\mu_0} \sum_{\alpha,\beta} F_{\alpha\beta} F^{\alpha\beta}$$

We also include an "interaction" term,

$$\mathcal{L}_I = A_\mu J^\mu$$

Recalling that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, our Euler-Lagrange equations become

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_I$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \sum_{\alpha=0}^{3} \frac{\partial}{\partial x^\alpha}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\mu)}\right] = 0$$

→ Reproduces Maxwell's equations.

In essence, we have just continued the process of reverse engineering to build this ... however, now that we've done it, we can take advantage of it to build other interactions.

In particular, the strong and weak forces are set up in a way that is explicitly designed to look just like E&M. A few details are changed - e.g., the weak fields are associated with a massive field - but are otherwise just like the field we already know and love.