Relativity and Gravity

Newton's law of gravitation,

$$\underset{\sim}{F}^{g}_{12} = - \frac{G m_1 m_2}{r^2} \underset{\sim}{e}_r$$

looks just like Coulomb's law for the force between two charges,

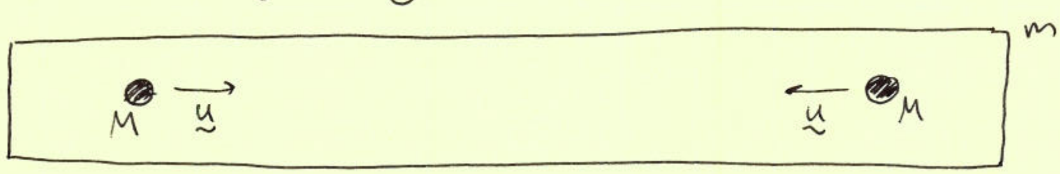
$$\underset{\sim}{F}^{q}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \underset{\sim}{e}_r$$

Without too much effort, we were able to promote our electric force law to a relativistically covariant form ... can we do the same for gravity?

Key to this working for E+M is that charges q_1 + q_2 are Lorentz invariants. Are the masses m_1 + m_2?

Yes... IF they represent rest mass. If that's the case, then gravity cannot act on light (rest mass is zero), and certain odd situations can occur.

Consider the following situation:



Box of mass m, 2 lumps of putty of rest mass M each. They zoom together with speed $|\underset{\sim}{u}| \simeq c$.

Prior to collision, the box feels a gravitational force

$$\underset{\sim}{F} = (m + 2M)\, g$$

If rest mass is all gravity cares about ... but after collision,

$$\underset{\sim}{F} = (m + 2\gamma(v)\, M)\, g\,.$$

If $|\underset{\sim}{v}| \simeq c$, this could be huge. Or, if one lump of putty is made of antimatter, then we'd have

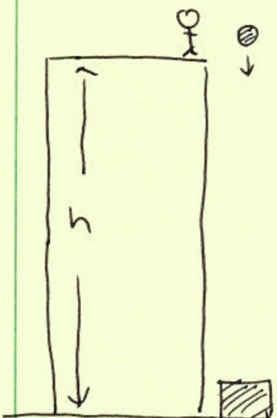
$$\underset{\sim}{F} = m g$$

after the collision - the two lumps of mass M would be converted into radiation.

So, if gravity couples to rest mass, it can lead to gravitational forces changing discontinuously ... which seems bad.

We choose instead to imagine that gravity couples to all forms of energy. In particular, it must affect light in some way: Light climbing out of a gravitational potential must lose energy.

Consider the following thought experiment, due to Einstein originally:



1. We stand on a tall building and drop a rock of mass m.
2. This rock enters a device after falling a height h. Its energy is

$$E = mc^2 + mgh.$$

3. The device converts the rock into a single photon of energy

$$h\nu = E = mc^2 + mgh$$

$\sqcap$ Pedantic side note: We cannot actually convert a single rock into a single photon: that would violate conservation of momentum. However, we can drop a rock and an "anti-rock" - a rock made of antimatter - and create two photons. The Earth can recoil and send both photons upward. $\sqcup$

4. When the photon reaches the top, convert it back to a rock. How much energy does this rock have?

To enforce energy conservation, the rock must have $E = mc^2$ - else we could make a perpetual motion machine out of this!

This means that the photon we shot up has $E' = h\nu' = mc^2$.

When climbing out of the gravitational field, the light is <u>redshifted</u>:

$$\frac{E'}{E} = \frac{E_{TOP}}{E_{BOTTOM}} = \frac{h\nu'}{h\nu} = \frac{mc^2}{mc^2 + mgh}$$

$$\rightarrow \quad \boxed{\frac{\nu'}{\nu} \simeq 1 - \frac{gh}{c^2}}$$

Worth looking at scale of the effect:

$$gh \simeq 100 \ m^2/s^2 \left(\frac{h}{10m}\right)$$

$$c^2 \simeq 10^{17} \ m^2/s^2$$

→ Frequency changes by a part in 10^{15} for every 10 meters of height change.

[[Note: More general form,

$$\frac{\nu'}{\nu} \simeq 1 - \frac{\Delta\Phi}{c^2}$$

where $\Delta\Phi \equiv$ change in gravitational potential between the two measurement points.]]

Move now from how an object responds to gravity to how gravity is generated.

You learned in previous classes that the gravitational potential arises from a mass density ρ_M as

$$\Phi(\underset{\sim}{r}) = -G\int \frac{\rho_M \, dV}{|\underset{\sim}{r} - \underset{\sim}{r}'|}$$

$$= -\frac{GM}{r} \quad \text{for a spherical mass } M.$$

This is equivalent to the differential form

$$\nabla^2 \Phi = -4\pi G \rho_M$$

How do we generalize this? Problem 1 is that ∇^2 is all the spatial derivatives: It picks out a preferred rest frame. So, let's replace it with $\Box$:

$$\nabla^2 \rightarrow \Box = \sum_{\mu,\nu} \eta^{\mu\nu} \partial_\mu \partial_\nu$$

Problem 2 is that the Newtonian law is given by a mass density, but energy seems to be the quantity that really matters. So let's replace matter density with energy density: $\rho_M \rightarrow \rho_E / c^2$.

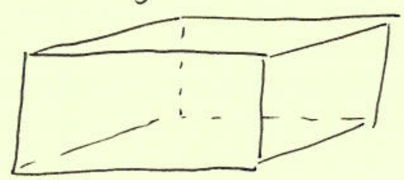
Proposal 1 for a relativistic law of gravity is then

$$\Box \Phi = -\frac{4\pi G}{c^2} \rho_E .$$

$\Box$ is a Lorentz invariant quantity; Φ is presumably a scalar. So ρ_E should be a scalar too.

Is it? Need to see how it transforms between frames.

Imagine a box filled with dust. Box is of volume V



Dust has mass M

Energy density in box's rest frame: $\rho_E = \frac{Mc^2}{V}$.

Now, from a frame in which box moves with relativistic speed:

→ Energy is boosted to $\gamma M c^2$

→ Volume is contracted to V/γ.

→ In new Lorentz frame, energy density is

$$\rho_E' = \frac{\gamma M c^2}{V/\gamma} = \gamma^2 \rho_E$$

Energy density is <u>NOT</u> a simple scalar! With this γ scaling, it's not even a component of a 4-vector. What is it?

Answer: ρ_E must be one component of a tensor!

Let's define this tensor in the "rest" frame to be

$$T^{\alpha\beta} \doteq \begin{bmatrix} \rho_E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Frame in which dust is at rest.}$$

To change frames, put $T^{\mu'\nu'} = \sum_{\alpha,\beta} \Lambda^{\mu'}{}_{\alpha} \Lambda^{\nu'}{}_{\beta} T^{\alpha\beta}$

For a general Lorentz transformation, $\underset{\sim}{v} = v^x \underset{\sim}{e}_x + v^y \underset{\sim}{e}_y + v^z \underset{\sim}{e}_z$, you'll find

$$T^{0'0'} = \gamma^2 \rho_E \rightarrow \text{Energy density in new frame}$$

$$T^{0'i'} = \gamma^2 \rho_E \frac{v^i}{c} = T^{i'0'} \qquad i = 1, 2, 3$$

→ Represents the flux of energy (divided by c) carried by dust in i-th direction -OR- Density of ith component of momentum of dust (times c)

$$T^{i'j'} = \gamma^2 \rho_E \frac{v^i v^j}{c^2} \qquad i, j = 1, 2, 3$$

→ Represents the flux of momentum component i in direction j.

The quantity we've just assembled is called the <u>stress-energy tensor</u>:

$$T^{\alpha\beta} = \text{flux of 4-momentum } p^\alpha \text{ in the } x^\beta \text{ direction.}$$

This gives us a covariant tool for describing how energy and momentum flow through spacetime in relativistic physics.

This is the mathematical tool that we will use to describe energy and momentum when it is distributed over a volume in relativity. The detailed form of the stress-energy tensor depends on the physics of the particular system under study. In all cases, the interpretation of the components in a given frame are as before:

$$T^{00} = \text{energy density in a frame}$$
$$T^{0i} = \text{energy flux -or- momentum density} = T^{i0}$$
$$T^{ij} = \text{momentum flux}$$

(All modulo factors of c.)

Important examples:

1. Dust, already discussed:

$$T^{00} = \gamma^2 \rho_0 \qquad T^{0i} = T^{i0} = \gamma^2 \rho_0 v^i/c \qquad T^{ij} = \gamma^2 \rho_0 \frac{v^i v^j}{c^2}$$

ρ_0 = energy density in "rest" frame.

Compact way to write this:

$$T^{\alpha\beta} = \rho_0 u^\alpha u^\beta / c^2$$

2. "Perfect" fluid:

$$T^{\alpha\beta} \doteq \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

→ In "rest" frame: no net flux of energy in fluid.

P = pressure

Key feature of a perfect fluid is its "spatial isotropy", i.e. the fact that $T^{xx} = T^{yy} = T^{zz}$ in the rest frame (and $T^{ij} = 0$ if $i \neq j$).

By boosting in another frame, can show that a simple way to write this in <u>any</u> frame is

$$T^{\alpha\beta} = (\rho + P) u^\alpha u^\beta + P \eta^{\alpha\beta}.$$