Recap: Relativity and gravity

- Determined that gravity should couple to energy rather than to rest mass. Choosing rest mass leads to weird situations like sudden changes in weight following collisions and annihilations.

CONSEQUENCE: Light must redshift as it climbs out of a gravitational potential!

$$\frac{\nu'}{\nu} = 1 - \frac{\Delta \Phi}{c^2} \simeq 1 - \frac{gh}{c^2} \quad \text{for } h_{\text{small}}.$$

This effect is observed constantly. Key part of the functioning of GPS.

- Source of gravity must also be energy; writing a possible gravitational field equation as

$$\Box \Phi = -4\pi G \rho_E / c^2$$

we determined that this has a big flaw: ρ_E is not a scalar, it is the component of a tensor - the stress-energy tensor.

Covariant description of energy and momentum distributed in spacetime, the stress-energy tensor:

$$T^{\alpha\beta} \equiv \text{flux of } p^\alpha \text{ in } x^\beta \text{ direction}$$

$T^{00} \equiv$ energy density in this frame

$T^{0i} \equiv$ energy flux in x^i direction — or

$\equiv$ density of momentum p^i

$= T^{i0}$

$T^{ij} \equiv$ flux of p^i in x^j direction

(modulo factors of c to be dimensionally correct.)

Examples:
$$T^{\alpha\beta} = \rho_0 \frac{u^\alpha u^\beta}{c^2} \doteq \begin{bmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ dust.

$$T^{\alpha\beta} = (\rho + P)\, u^\alpha u^\beta / c^2 + P \eta^{\alpha\beta}$$
$$\doteq \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

"Perfect" fluid.

Purpose of stress-energy tensor: Gives us a covariant tool for formulating the conservation of energy and momentum

In a given frame, 3-momentum $\underset{\sim}{p}$ and energy E are conserved. For particles, this generalizes to conservation of 4-momentum. How about for momentum and energy distributed over a region of spacetime?

Consider some volume $\mathcal{V}$ with a surface $\mathcal{S}$. The energy in this region is described by a density $\rho_E \equiv T^{00}$, and its flux by $\mathcal{E}^i = c\, T^{0i}$.

Conservation of energy in this volume is given by

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho_E \, dV}_{\text{Rate of change of energy in } \mathcal{V}} + \underbrace{\int_{\mathcal{S}} \underset{\sim}{\mathcal{E}} \cdot d\underset{\sim}{A}}_{\text{Flux of energy through the surface } \mathcal{S}.} = 0$$

Using the divergence theorem, we can rewrite this as

$$\int_{\mathcal{V}} \left(\frac{\partial \rho_E}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{\mathcal{E}} \right) dV = 0$$

-or-

$$\int_{\mathcal{V}} \left(\frac{\partial T^{00}}{\partial t} + c \sum_{i=1}^{3} \frac{\partial}{\partial x^i} T^{0i} \right) dV = 0$$

-or

$$\int_{\mathcal{V}} \sum_{\alpha=0}^{3} \left(\frac{\partial}{\partial x^\alpha} T^{0\alpha} \right) dV = 0$$

$$\rightarrow \quad \sum_{\alpha=0}^{3} \boxed{\partial_\alpha T^{0\alpha} = 0}$$

Same game for momentum:

Let P^i = density of momentum p^i

$\equiv T^{i0}/c$

and let Π^{ij} = flux of p^i in x^j direction

$\equiv T^{ij}$

Conservation of momentum is then expressed by the rule

$$\frac{\partial}{\partial t}\int_{\mathcal{V}} P^i \, dV + \int_{\mathcal{S}} \sum_j \Pi^{ij} \, dA^j = 0$$

Rate of change of momentum component p^i in volume $\mathcal{V}$

Flux of p^i through surface $\mathcal{S}$

We again apply divergence theorem:

$$\int_{\mathcal{V}} \left[\frac{\partial P^i}{\partial t} + \sum_{j=1}^{3} \frac{\partial \Pi^{ij}}{\partial x^j} \right] dV = 0$$

-or-

$$\int_{\mathcal{V}} \left[\frac{1}{c}\frac{\partial T^{i0}}{\partial t} + \sum_{j=1}^{3} \frac{\partial T^{ij}}{\partial x^j} \right] dV = 0$$

-or-

$$\int_{\mathcal{V}} \left[\sum_{\alpha=0}^{3} \frac{\partial T^{i\alpha}}{\partial x^\alpha} \right] dV = 0$$

$$\rightarrow \sum_{\alpha=0}^{3} \boxed{\partial_\alpha T^{i\alpha} = 0}$$

Combine these two rules, and we have

$$\boxed{\sum_{\alpha=0}^{3} \partial_\alpha T^{\beta\alpha} = 0}$$

This is a fully covariant expression for energy and momentum conservation in relativity.

Interlude

We now have all the pieces we need to begin building a theory of gravity that incorporates the principles of relativity. Let's recap the things we've figured out so far:

- Gravity must couple to "energy." Light is redshifted by gravity.

- Energy density cannot be the source of gravity - it's only one component of a geometric object, the stress-energy tensor.
  - → Recall we try to build our laws of physics around geometric objects as much as possible. Means that the stress-energy tensor itself is our best candidate for a source to gravity.

- Unaccelerated motion in special relativity leads to motion which extremizes aging along the trajectory. Trajectories which do this are straight lines in spacetime: Geodesics.

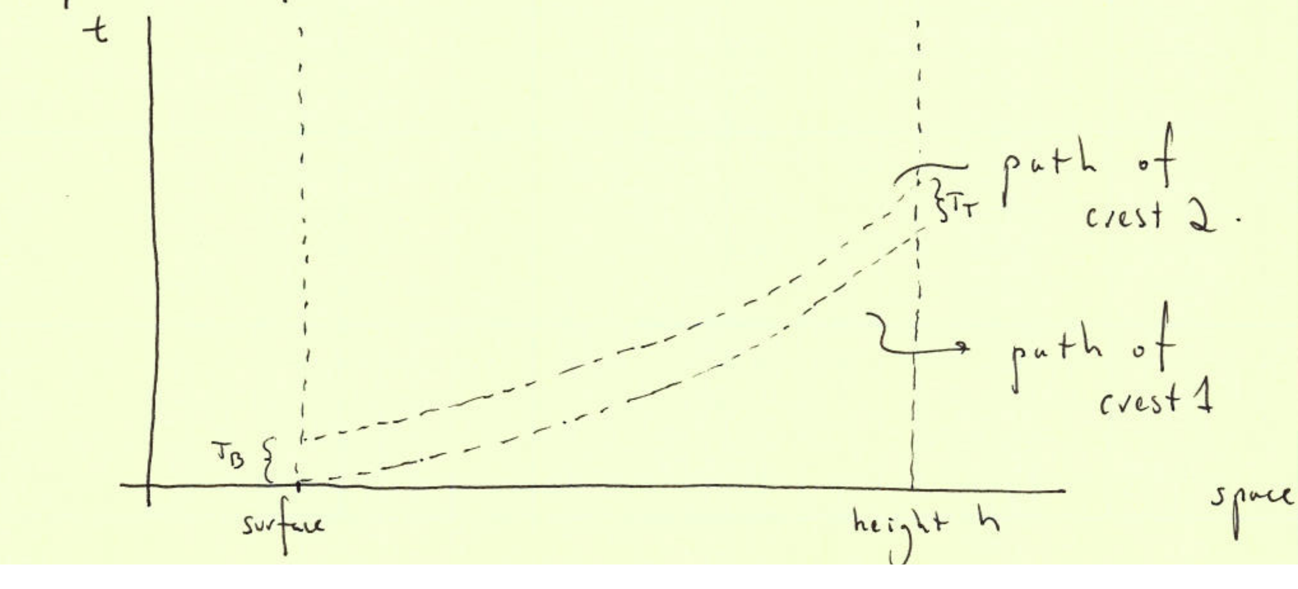
To put all these pieces together, let's start thinking about what is so "special" about special relativity.

The key thing we come back to, again and again is the notion of a Lorentz frame: A frame of reference in which things move at constant velocity if no forces act on them. It is an inertial frame; we move between different Lorentz frames by a Lorentz transformation.

Special relativity assumes we can "cover" all of spacetime - all times, all places - using a single Lorentz frame. I.e., special relativity says that a global Lorentz frame makes sense.

Gravity breaks this. When gravity is included, we cannot have a global Lorentz frame covering all of spacetime.

Consider a radio pulse propagating from the surface of the Earth to a height h. Examine paths through spacetime of two successive crests of radio wave:



We don't yet know how gravity will affect the pulse's path through spacetime, so we imagine it is bent somehow. This gives us the path of crest 1.

How do we then draw crest 2?

If we require spacetime to be Lorentz everywhere, then there is nothing special about time or place. The path of the 2ND crest is identical to the path of the 1ST one, just shifted later in time.

→ If this is true, then the wave period at the bottom is the same as the wave period at the top: $T_B = T_T$

Which implies that the frequency is the same at the top and bottom: $\nu_B = \nu_T$ since $\nu = \frac{2\pi}{T}$

→ WHICH CONTRADICTS THE REDSHIFT RESULT:

$$\frac{\nu_B}{\nu_T} = 1 - \frac{gh}{c^2}$$

Our starting assumption cannot be correct:

In the presence of gravity, we cannot have global Lorentz reference frames.

This sounds bad! The Lorentz frame was our key tool for everything we've done up til now.

Before giving up, think carefully about what an inertial frame has meant: In the absence of external forces, objects maintain their relative velocities.

→ The same thing happens in a "Freely Falling Frame": All objects feel the same acceleration due to gravity; so, in the FFF, gravity's influence cancels out.

New rule: In the absence of non-gravitational forces, objects maintain their relative velocities in a FFF.

Key to this working: the equivalence of "gravitational mass" with "inertial mass": $mg = ma \rightarrow a = g$ for all m.

As summarized by Einstein, cannot distinguish gravity and uniform acceleration.

Actually, gravity is never uniform: some variation leads to tides. Trajectories which are initially parallel in spacetime will be focused or diverge due to tidal forces - spacetime must have curvature.

We will work with two ideas to try to rescue relativity with gravity:

1: In a Freely Falling Frame, things look inertial, and so we expect that Lorentz frames describe spacetime <u>over</u> a <u>finite</u> <u>region</u>: "Local Lorentz Frame"

2: The size of that finite region is controlled by gravity's tides: Stronger tides means that the size of the region well-described by a Lorentz frame is smaller.

These ideas are used to formulate the

<u>Principle of Equivalence</u>.

Two variants we'll use:

W: Over sufficiently small regions, the motion of freely falling bodies due to gravity cannot be distinguished from uniform acceleration.

"Weak Equivalence Principle"

E: In sufficiently small regions of spacetime, the laws of physics in a freely falling frame reduce to those of special relativity

"Einstein Equivalence Principle"

The Einstein equivalence principle gives us the tool needed to compute motion in relativistic gravity. Here's the logic:

1: In Freely Falling Frame, motion equivalent to that of special relativity.

2: S.R. motion is a "straight line" ...

3: which is a geodesic!

→ Motion in a general spacetime is described by computing geodesics in a general spacetime metric:

$$ds^2 = \sum_{\alpha, \beta} g_{\alpha\beta}\, dx^\alpha dx^\beta$$

<u>not</u> just diag $(-1, 1, 1, 1)$

Two things to do:

- Get spacetime for a given gravitational configuration
- Describe geodesics in general spacetime.