Recap:

- Reviewed stress-energy tensor. Showed that it can be used to give a covariant formulation of energy and momentum conservation:

$$\sum_{\alpha=0}^{3} \partial_\alpha T^{\alpha\beta} = 0$$

- Argued that gravity means we can no longer have "global" Lorentz frames. "Global" Lorentz frame means using the metric $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ everywhere/everywhen; if we move away from this, the spacetime metric becomes $g_{\alpha\beta}$, a tensor whose components are functions of position in space and time.

- Guidelines for moving forward provided by the principle of equivalence:

  Over sufficiently small regions, the motion of freely falling bodies due to gravity cannot be distinguished from uniform acceleration.

  In sufficiently small regions of spacetime, the laws of physics in a freely falling frame reduce to those of special relativity.

Two big tasks:

1. Understand how spacetimes $g_{\mu\nu}$ arise given an arrangement of stress energy $T^{\mu\nu}$.

2. Understand how, given a spacetime, matter moves around, ie how to build geodesics.

"Matter tells spacetime how to curve; spacetime tells matter how to move." - John A. Wheeler.

Item 1: Will do in a somewhat schematic manner in next lecture. Doing this with rigor beyond the scope of 8.033!

Item 2: "Easily" done.

Key idea: Develop your intuition in freely falling frame (FFF). How do bodies move there? Simple: If unaccelerated, they must follow geodesics.

Equivalence principle tells us that if we can formulate the motion in the FFF, we've formulated motion in all frames:

→ Motion in a general spacetime is described by a geodesic if the body feels no forces (aside from gravity).

Geodesics: Previously, we examined geodesics in the following way:

$$c\Delta\tau = \int_A^B \sqrt{-\eta_{\alpha\beta}\, dx^\alpha\, dx^\beta}$$: Accumulated proper time for a time like observer

We then chose to single out the coordinate t as special, and wrote

$$\Delta\tau = \int_A^B \left[1 - \frac{1}{c^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right]^{1/2} dt$$

where $\dot{x} \equiv dx/dt$, etc. Defining $f \equiv \left[1 - \frac{1}{c^2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right]^{1/2}$, we apply Euler equations,

$$\frac{\partial f}{\partial q} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}} = 0$$

and find straight line solutions:

$$x = \dot{x}t + x_0 \quad , \quad \text{etc for } y, z.$$

We want to redo this, fixing up two "flaws":

1. Don't want to pick out time as a special coordinate: picks out some frame as special.
2. We want to handle a spacetime metric that is not constant.

Dealing with the second point isn't so hard - we've mentioned this issue in passing by virtue of the fact that we can do special relativity in "non-inertial" coordinates:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{"Inertial"}$$
$$= -c^2dt^2 + dr^2 + r^2d\phi^2 + dz^2 \qquad \text{"Cylindrical"}$$
$$= -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2 \qquad \text{"Spherical"}$$

We will reserve the symbol $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ for the metric in inertial coordinates. We will use $g_{\alpha\beta}$ to denote the metric in general:

$$ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta}\, dx^\alpha dx^\beta$$

Cylindrical:

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x^0 = ct \qquad x^1 = r$$
$$x^2 = \phi \qquad x^3 = z$$

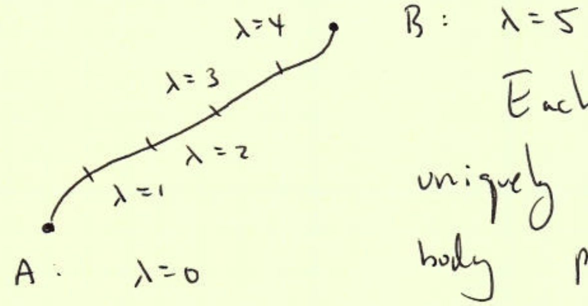
Spherical:

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$$x^0 = ct \qquad x^1 = r$$
$$x^2 = \theta \qquad x^3 = \phi$$

Next, we want to formulate rules describing geodesic in a way that does not pick out coordinate t as "special."

Let's imagine that there's some parameter λ that increases monotonically along the trajectory:



Each value of λ maps uniquely to some event that the body passes through on the trajectory.

The accumulated proper time on the trajectory is

$$\Delta\tau = \frac{1}{c}\int_A^B \sqrt{-g_{\alpha\beta}\, dx^\alpha\, dx^\beta}$$

$$= \frac{1}{c}\int_A^B \sqrt{-g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}\; d\lambda$$

where $\dot{x}^\mu \equiv \dfrac{dx^\mu}{d\lambda}$

Now, let's extremize this in the normal way, using x^μ and $\dot{x}^\mu$ as our variational quantities.

Actually, before jumping into this, it is useful to massage a bit first.

Define $\quad f \equiv g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu$

So $\quad \Delta\tau \equiv \dfrac{1}{c}\displaystyle\int_A^B (-f)^{1/2}\, d\lambda$

As before, we imagine that there is some solution that gives us the extremum, and consider a variation:

$$x^\mu(\lambda) = x^\mu_e(\lambda) + \alpha A^\mu$$

Extremal trajectory

Deviation that vanishes at endpoints of integral

Extremum of $\Delta\tau$ defined by

$$\frac{\partial(\Delta\tau)}{\partial\alpha} = \frac{1}{c}\int_A^B \frac{\partial}{\partial\alpha}(-f)^{1/2}\,d\lambda = 0$$

$$\text{-or-} \quad = -\frac{1}{2c}\int_A^B \frac{\partial f/\partial\alpha}{[-f]^{1/2}}\,d\lambda = 0$$

We can now simplify this with a clever choice for λ. Recall we defined λ simply by demanding that it increase monotonically from event A to event B.

Perfect choice for this: let λ simply be proper time experienced by the body!

With this choice, $-f = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

$$= -\vec{u}\cdot\vec{u}$$

$$= c^2$$

$\rightarrow$ the f in the denominator becomes a constant!

Our condition is then

$$\frac{\partial(\Delta\tau)}{\partial\alpha} \propto \int_A^B \frac{\partial f}{\partial \alpha}\, d\lambda = 0$$

This means the integral we want to extremize is simply

$$S = \int_A^B L\, d\tau$$

where $L = \frac{1}{2} f = \frac{1}{2}\sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

$$\equiv \frac{1}{2}\sum_{\mu,\nu} g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu$$

is a <u>Lagrangian</u> for relativistic motion! (Really, a Lagrangian per unit mass.)

When we apply Euler to this,

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\alpha} = 0$$

we get equations governing the geodesics of the spacetime $g_{\mu\nu}$.

In all of the examples we will study in 8.033, we will build L for a given spacetime, and then build Euler-Lagrange equations to describe motion in this spacetime. Example: Newtonian limit on pset #7.

HOWEVER, it is useful to know that we in fact get an interesting result if we leave the metric totally unspecified: $L = \frac{1}{2} \sum g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ leads to

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \sum_{\alpha,\beta} \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0}$$

where

$$\boxed{\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} \sum_{\gamma=0}^{3} g^{\mu\gamma} \left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \right)}$$

The quantity $\Gamma^\mu{}_{\alpha\beta}$ is called a "connection coefficient" or "Christoffel symbol," and describes the manner in which the basis vectors, $\vec{e}_\mu$, vary with position: It connects $\vec{e}_\mu(\vec{x})$ with $\vec{e}_\mu(\vec{x} + \Delta\vec{x})$.

A few points worth noting:

1. If $\partial_\gamma g_{\alpha\beta} = 0$, then

$$\frac{d^2 x^\mu}{d\tau^2} = 0 \rightarrow \text{A straight line!}$$

2. Suppose we defined our trajectory in a slightly different way: As we moved along the trajectory, imagine that we require our tangent to the trajectory at step N be parallel to the tangent at step N+1.

On blackboard, this defines a straight line.

On a curved surface, this defines a trajectory that is as straight as possible: A geodesic! This is an alternate way to derive the geodesic equation, and gives us another interpretation of its meaning:

> A geodesic defines the trajectory that is as straight as possible given the geometry on which the trajectory resides.

3. We will never use this general form in 8.033!