Recap:

- Geodesics in greater generality: Found by developing Euler-Lagrange equations, $\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\alpha} = 0$, for the Lagrangian $L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ (where $\dot{x}^\mu = dx^\mu / d\tau$).

- If you do this for a totally general metric, then geodesics given by the equation

$$\frac{d^2 x^\mu}{d\tau^2} + \sum_{\alpha,\beta} \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

where $$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\sum_{\gamma=0}^{3} g^{\mu\gamma}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right)$$

- This also defines "parallel transport": The trajectory we get by moving a vector parallel to itself in some direction in spacetime.

Where do our spacetimes come from? Very, very schematic discussion of the Einstein field equation of GR.

Intuition: Consider Newton's field equation:

$$\nabla^2 \Phi = 4\pi G \rho_{N}$$

Two derivs of potential = 1 deriv of grav. acceleration = <u>TIDAL</u> field

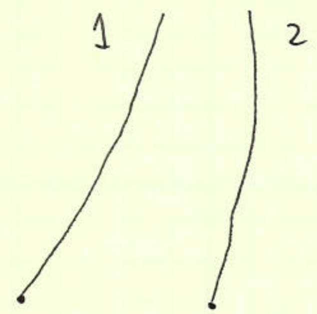
Source.

Newton: "Tides" = "Matter density"

Einstein: Matter density $\longrightarrow T^{\alpha\beta}$. Expect

"Tides" = "stress-energy"

So, how do we make "tides" vigorous?
As we've already discussed, tides are related to a spacetime's curvature: They describe how parallel trajectories in spacetime converge or diverge. So, we just need to quantify that convergence or divergence.



Evaluated on path 1

Geodesic 1: $\dfrac{d^2 x^\mu}{d\tau^2} + \sum_{\alpha,\beta} \Gamma^\mu{}_{\alpha\beta} \dfrac{dx^\alpha}{d\tau} \dfrac{dx^\beta}{d\tau}$

Geodesic 2: Same thing, but now evaluated on path 2.

Finally, let $\delta x^\mu \equiv x^\mu_2 - x^\mu_1$. Then, with some effort, we can show that

$$\frac{D^2(\delta x^\mu)}{D\tau^2} = R^\mu{}_{\alpha\beta\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \delta x^\nu$$

where $\dfrac{D}{D\tau}$ is a "covariant" derivative, one that accounts for the curvature of the metric in careful way, and where

$$R^\mu{}_{\alpha\beta\nu} = \partial_\beta \Gamma^\mu_{\nu\alpha} - \partial_\nu \Gamma^\mu_{\beta\alpha} + \sum_{\gamma=0}^{3} \left(\Gamma^\mu_{\beta\gamma} \Gamma^\gamma_{\nu\alpha} - \Gamma^\mu_{\nu\gamma} \Gamma^\gamma_{\beta\alpha} \right)$$

is the "Riemann curvature tensor."

A few points to note:

1. It has terms of the form $\partial \Gamma$, which is itself of the form ∂(metric). Hence,

$$\text{curvature} \sim \partial^2 (\text{metric})$$

- Just like curvature of a curve in 1-D is related to its 2^{ND} derivative
- Just as tidal force looks like 2 derivs of potential.

2. It has terms of the form $\Gamma^2 \sim (\partial \text{metric})^2$
   → Non linear!

3. It has 4 indices – naively, it has $4^4 = 256$ components!

Symmetries in fact reduce this: Only 20 of those 256 are independent. Still, bit of a mess to deal with.

Following our earlier logic, we expect our gravity field equation to be of the form

$$(\text{Riemann}) = k\, T^{\mu\nu}$$

where k is some constant needed to get dimensions right. However, Riemann has 4 indices, stress-energy only 2.

Solution: Use metric to combine two of the indices, called the trace:

$$R^{\mu\nu} = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g_{\alpha\beta} R^{\alpha\mu\beta\nu}$$

Connect 1ST + 3RD indices.

Attempt 2:

$$R^{\mu\nu} = k\, T^{\mu\nu}$$

Does this work?

Not quite. Recall special relativity that $\sum_\mu \partial_\mu T^{\mu\nu} = 0$. A curved spacetime variant also holds:

$$\sum_{\mu=0}^{3} \nabla_\mu T^{\mu\nu} = 0$$

"Covariant" analog of ∂_μ, accounts for curvature of spacetime.

Unfortunately, $\sum_{\mu=0}^{3} \nabla_\mu R^{\mu\nu} \neq 0$. However, it is not difficult to show that

$$\sum_{\mu=0}^{3} \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 ,$$

where $R = \sum_{\alpha,\beta} g_{\alpha\beta} R^{\alpha\beta}$.

Definition: $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ "Einstein curvature"

Our candidate equation is then

$$G^{\mu\nu} = k T^{\mu\nu}$$

If $k = 8\pi G / c^4$, then in an appropriate limit, this reproduces Newtonian gravity:

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

The Einstein field equation of general relativity.

Note general structure:

"Big nonlinear coupled mess of metric derivatives" = "Flow of energy and momentum"

Some important solutions:

1. $g_{00} = -(1+2\Phi)$

$g_{11} = g_{22} = g_{33} = 1 - 2\Phi$

$x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z$

$\Phi = -\frac{GM}{rc^2}, \quad r = \sqrt{x^2+y^2+z^2}$

Corresponds to a spherical body of mass M for $\Phi \ll 1$ ... reproduces Newtonian gravity.

2. $g_{00} = -(1 - 2GM/rc^2)$

$g_{11} = (1 - 2GM/rc^2)^{-1}$

$g_{22} = r^2 \quad g_{33} = r^2 \sin^2\theta$

$x^0 = ct \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi$

The Schwarzschild metric: Exact solution outside any spherical distribution of mass M.

If body satisfies $T^{\mu\nu} = 0$ everywhere except at $r=0$, then this represents a non-spinning black hole.

"Schwarzschild metric" – Karl Schwarzschild, 1916.

3. $$g_{00} = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} \qquad g_{11} = \frac{\Sigma}{\Delta} \qquad g_{22} = \Sigma$$

$$g_{33} = \left(\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\right)\sin^2\theta$$

$$g_{03} = g_{30} = -\frac{2a\tilde{M}r\sin^2\theta}{\Sigma}$$

"Kerr metric"
1963

$$\Delta = r^2 - 2\tilde{M}r + a^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\tilde{M} = \frac{GM}{c^2} \qquad a = \frac{J}{Mc}$$

$$x^0 = ct \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi$$

Represents spacetime of a black hole with mass M and spin angular momentum J. "Kerr metric"

4. $$g_{00} = 1 \qquad g_{11} = a^2(t)$$

$$g_{22} = a^2(t)\, r^2$$

$$g_{33} = a^2(t)\, r^2\sin^2\theta$$

Function $a(t)$ determined by density ρ, pressure P:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3c^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P)$$

Describes a universe full of perfect fluid with density ρ and pressure P.

"Friedman-Robertson-Walker metric"

Example calculations: Newtonian metric.

Suppose we have an observer who is at rest in the "Newtonian" spacetime : $dx/d\tau = dy/d\tau = dz/d\tau = 0$. What is $dt/d\tau$?

We still require $\vec{u}\cdot\vec{u} = -c^2$. Why? If we go into a freely falling frame, the calculation is identical to special relativity. Invariance of inner product tells us that the result holds in general.

$$\vec{u}\cdot\vec{u} = \sum g_{\alpha\beta}\, u^\alpha u^\beta$$

$$= -(1+2\Phi)\, c^2 \left(\frac{dt}{d\tau}\right)^2 + 0 \quad = \quad -c^2$$

$$\rightarrow \quad \frac{dt}{d\tau} = \left(1+2\Phi\right)^{-1/2} \simeq (1-\Phi) \qquad (\text{since } \Phi \ll 1)$$

$$\simeq 1 + \frac{GM}{rc^2} .$$

Consider observers at two different heights:

$$\frac{(dt/d\tau)_1}{(dt/d\tau)_2} = \frac{1 + GM/r_1c^2}{1 + GM/r_2c^2} \simeq 1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= 1 - \frac{\Delta\Phi_N}{c^2}$$

→ Same result we obtained for ticking of clocks by considering light redshift.

These metrics give us a relativistic way to compute the effect of gravity. Involves a big change in philosophy:

Newton: Gravity is a <u>force</u> : $\underset{\sim}{F}_g = -\frac{GMm}{r^2}\,\underset{\sim}{e}_r = m\underset{\sim}{a}$.

Einstein: Bodies follow geodesics of spacetime. If spacetime is "weakly curved," then the trajectory for slow motions is identical to what the Newtonian gravitational force predicts.

Score card:

1. Reproduces Newton when $GM/rc^2 \ll 1$, and for non-relativistic motions ($|\underset{\sim}{v}| \ll c$).
   → If it had not, it would not have been a viable theory!

2. Predicts clocks run at different rates, and redshifting of light in gravity.
   → Measured every day. Key component of high precision position measurements.

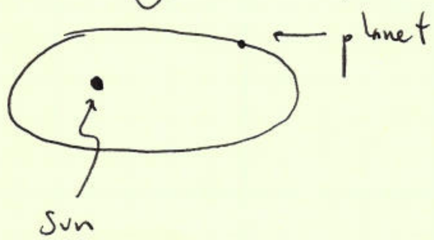
3. Bending of light by gravity
   Weak, but measurable effect in solar system: Eddington measured it in 1919. Can now measure with part per million accuracy!
   We now exploit this effect: Measure light bending, infer how much mass/energy is present: Gravitational lensing.

4. Corrections to Newtonian motion

On pset, you examined the limit $\Phi \ll 1$ and $|\underset{\sim}{v}| \ll c$. Geodesics in this limit reproduce the Newtonian force law:

$$\frac{d^2 \underset{\sim}{x}}{dt^2} = -\frac{GM}{r^3}\underset{\sim}{x}$$

Newton's gravity predicts orbits that are closed ellipses:



Ellipticity highly exaggerated...

Observations the centuries show that the ellipses don't quite close - instead they precess. The axes slowly rotate around the sun:

Most of this effect is due to interactions with other planets. However, there is a small residual which could not be accounted for.

Largest residual for Mercury: Ellipse precesses at 43 arcseconds per century.

Note: 1 degree = 3600 arcseconds.

Repeat exercise from pset #7, but now keep 1ST corrections to this leading solution (i.e. don't discard terms of order v^2/c^2).

Result:
$$\frac{d^2 \underset{\sim}{x}}{dt^2} = -\frac{GM\underset{\sim}{x}}{r^3} - \frac{GM}{r^3}\frac{|v|^2}{c^2}\underset{\sim}{x} + \frac{4GM(\underset{\sim}{x}\cdot\underset{\sim}{v})\underset{\sim}{v}}{c^2 r^3}$$

This force gives an ellipse that precesses. Rate of precession we find for Mercury is

$$\frac{d\phi_{peri}}{dt} = \frac{6\pi G M_\odot}{a(1-e^2)Pc^2}$$

$M_\odot$ = mass of sun = 1.99×10^{30} kg

a = semi-major axis of Mercury's orbit
= 57.9×10^6 km

e = eccentricity of Mercury's orbit
= 0.2

P = Period of Mercury's orbit
= 88 days

$\rightarrow$ $\boxed{\dfrac{d\phi_{peri}}{dt} = 42.9 \text{ arcseconds per century.}}$