Recap:

- Sketched the Einstein field equation

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Big nonlinear derivatives acting on spacetime metric: describes tides of gravity

Flow of energy and momentum in spacetime

- Introduced several exact and approximate solutions.

- Reviewed Big tests of general relativity:

0. Newtonian limit
1. Redshifting of light
2. Bending of light
3. Explanation of Mercury's perihelion precession.

Quick summary of perihelion precession (rushed through in the previous class):

Repeat calculation of pset #7, but don't discard all velocity terms:

$$\frac{d^2 \underset{\sim}{x}}{dt^2} = -\frac{GM\underset{\sim}{x}}{r^3}\left(1 + \frac{|v|^2}{c^2}\right) + \frac{4GM}{r^3}\frac{(\underset{\sim}{x}\cdot\underset{\sim}{v})\underset{\sim}{v}}{c^2}$$

A solution for this force law is an elliptical orbit that slowly precesses, with frequency

$$\frac{d\phi_{peri}}{dt} = \frac{6\pi G M_\odot}{a(1-e^2)Pc^2}$$

$M_\odot$ = mass of sun = 2×10^{30} kg

a = semimajor axis of Mercury's orbit

= 57.9×10^6 km

e = eccentricity of Mercury's orbit

= 0.2

P = period of Mercury's orbit

= 88 days

$$\left(\frac{d\phi_{peri}}{dt}\right)_{GR} = 42.98 \pm 0.04 \ \frac{\text{arcsec}}{\text{century}}$$

$$\left(\frac{d\phi_{peri}}{dt}\right)_{observed} = 42.5 \pm 0.6 \ \frac{\text{arcsec}}{\text{century}}$$

Assessment

Does all this mean that general relativity is "the" theory of relativistic gravity?

Certainly not! I hope you noticed that several decisions we made in sketching the Einstein field equation were kind of ad hoc. Two examples:

1. We selected the left-hand side by demanding that it be some symmetric 2^{ND} rank tensor with vanishing divergence. Is $G^{\mu\nu}$ the only such choice?

   → no.

2. We chose the constant $k = 8\pi G / c^4$ by requiring that this theory reproduce Newton's gravity. Does this k have to be a constant?

   → no.

General relativity is, in a quantifiable sense, the simplest theory of relativistic gravity. It passes all the tests ... but we've really only tested it for "weak gravity."

How does it hold up for strong gravity situations? What does "strong gravity" mean?

Weak gravity means $GM/rc^2 \ll 1$. If this is not the case, then we turn to the exact Schwarzschild solution:

$$g_{00} = -\left(1 - \frac{2GM}{rc^2}\right)$$

$$g_{11} = \left(1 - \frac{2GM}{rc^2}\right)^{-1}$$

$$g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta$$

$$x^0 = ct \qquad x^1 = r \qquad x^2 = \theta \qquad x^3 = \phi$$

An exact solution with $T^{\mu\nu} = 0$.

A lot of what we've learned about relativity comes from examining clocks and light - start here.

Clocks: Consider a static observer, $u^r = u^\theta = u^\phi = 0$.

$$\vec{u} \cdot \vec{u} = -c^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2$$

$$\rightarrow \frac{dt}{d\tau} = 1 \Big/ \sqrt{1 - \frac{2GM}{rc^2}}$$

Very far away, $dt/d\tau \simeq 1$: The coordinate t is equivalent to proper time of a very distant observer.

But, not far away, $d\tau = dt\sqrt{1 - \frac{2GM}{rc^2}}$

$\rightarrow$ Clocks tick slowly at finite r

<u>AND</u> <u>STOP</u> at $r = 2GM/c^2$!

Next, imagine light on a radial trajectory:

$$\vec{p}\cdot\vec{p} = 0$$

$$\rightarrow \quad -c^2\left(\frac{dt}{d\lambda}\right)^2\left(1-\frac{2GM}{rc^2}\right) + \left(\frac{dr}{d\lambda}\right)^2\left(1-\frac{2GM}{rc^2}\right)^{-1} = 0$$

$$\rightarrow \quad \frac{dr}{dt} = \pm\, c\left(1-\frac{2GM}{rc^2}\right)$$

Speed of light in coordinate system is less than c! In fact, it goes to zero as $r \rightarrow 2GM/c^2$.

Next, consider energy of light emitted from $r=R$, but measured infinitely far away:

$$\frac{E(r\rightarrow\infty)}{E(r=R)} = \frac{-\vec{p}\cdot\vec{u}|_\infty}{-\vec{p}\cdot\vec{u}|_R}$$

$$\vec{p}\cdot\vec{u} = p_t\left(1-\frac{2GM}{rc^2}\right)^{-1/2}$$

p_t = constant, since the metric + ~~th~~ hence the Lagrangian are independent of time.

$$\rightarrow \quad \frac{E(r\rightarrow\infty)}{E(r=R)} = \sqrt{1-\frac{2GM}{Rc^2}}$$

Notice: if $R = \frac{2GM}{c^2}$, $E(r\rightarrow\infty) = 0$.

<u>ALL</u> energy at that radius is drained away by gravitational redshift!

Nothing at $r = 2GM/c^2$ can communicate with the rest of the universe: Anything "behind" that radius is causally disconnected by this "Event horizon."

Any object so compact that it has an event horizon is a "black hole" in spacetime.

Now, what happens for a material body (not light) moving in this spacetime? Assemble Lagrangian:

$$L = \frac{1}{2} \sum g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$= \frac{1}{2} \left[-c^2 \left(1 - \frac{2GM}{rc^2}\right) \dot{t}^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right]$$

You show on pset #9, $\partial L/\partial t = 0$, $\partial L/\partial \phi = 0$

$$\rightarrow \quad \hat{E} = -\frac{\partial L}{\partial \dot{t}} = c^2 \left(1 - \frac{2GM}{rc^2}\right) \frac{dt}{d\tau} = \text{const}$$

$$\hat{L}_z = r^2 \sin^2\theta \, \frac{d\phi}{dt} = \frac{\partial L}{\partial \dot{\phi}} = \text{const}$$

Combine with $L = \vec{u}\cdot\vec{u}/2 = -c^2/2$, we can turn L plus definitions of $\hat{E}$ + $\hat{L}_z$ into an equation of motion for r: $\left(\frac{dr}{d\tau}\right)^2 = \frac{\hat{E}^2}{c^2} - V_{eff}$

$$V_{eff} = \left(1 - \frac{2GM}{rc^2}\right)\left(c^2 + \frac{\hat{L}_z^2}{r^2}\right)$$

(Note: using $\theta = \pi/2$ + $\dot{\theta} = 0$ at $t = 0$ to get $\ddot{\theta} = 0 \rightarrow$ $\theta = \pi/2$ for all time.)

Example 1: Consider an object with $\hat{L}_z = 0$, and

$$\hat{E} = c^2 \sqrt{1 - \frac{2GM}{c^2 R}}$$

$$\rightarrow \left(\frac{dr}{d\tau}\right)^2 = 2GM \left[\frac{1}{r} - \frac{1}{R} \right]$$

Can easily solve for $\tau(r)$:

$$\tau = \frac{4GM}{3c^3} \left[\left(\frac{Rc^2}{2GM}\right)^{3/2} - \left(\frac{rc^2}{2GM}\right)^{3/2} \right]$$

This is motion as a function of proper time: Someone falling in reaches $r = 0$ in finite proper time. What happens as we cross $r = 2GM/c^2$? Nothing special!

What if we change our time variable? τ is time of the infalling body itself ... t is time as measured very far away.

$$\frac{dr}{dt} = \frac{dr}{d\tau} \left(\frac{dt}{d\tau}\right)^{-1}$$

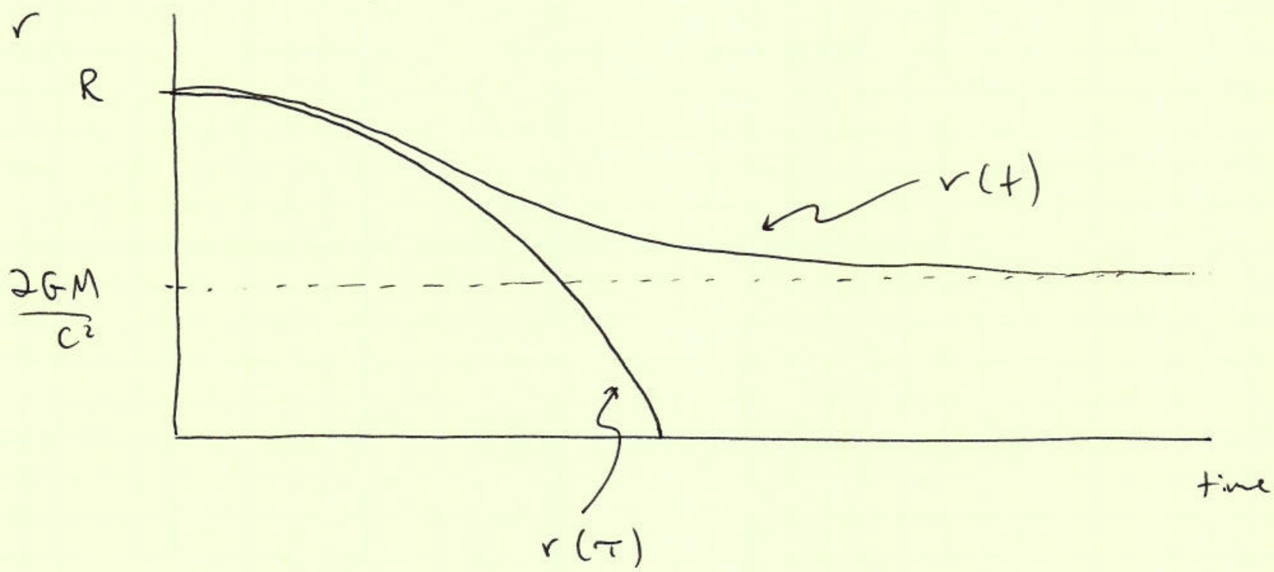
$$= \frac{\sqrt{2GM\left(\frac{1}{r} - \frac{1}{R}\right)} \left(1 - \frac{2GM}{rc^2}\right)}{\hat{E}/c^2}$$

Solution:

$$t = \frac{2GM}{c^3}\left\{\left[\ln\left(\frac{(c^2r/2GM)^{1/2}+1}{(c^2r/2GM)^{1/2}-1}\right) - 2\sqrt{\frac{c^2r}{GM}}\left(1+\frac{c^2r}{6GM}\right)\right] - \left[\text{same thing with } r\to R\right]\right\}$$

This <u>DIVERGES</u> as $r \to 2GM/c^2$!

Two views of radial infall:



Drastically different views! Can they be reconciled?

Key is that the behavior of light at $r = 2GM/c^2$ implies pathological behavior of clocks - the coordinate t - in this spacetime. The distant observer can never see the infalling body cross the horizon since light from there could never reach large radius. Since clocks run slow, motion - a kind of clock also runs slow.