The value of k and the behavior of $a(t)$ must be determined from the Einstein field equations, which we'll do soon. As a setup, it is useful to examine how light and matter behave in this spacetime.

Begin by asking what happens to observers at rest:

$$u^t = c, \quad u^{\bar{r}} = u^\theta = u^\phi = 0.$$

Examine geodesics: Find that they remain fixed at coordinate $(\bar{r}, \theta, \phi)$.

However, as they are fixed there, the proper separation of different observers changes at $a(t)$ evolves. These observers "comove" as the universe's geometry changes – they are "comoving" observers.

Next, examine light – Crucial, since we use it to measure and understand our universe.

For simplicity, focus on $k=0$. (Calculation can be generalized to $k = \pm 1$, but it is messy.)

Imagine that light is emitted at some time t_e, and received ~~###~~ by an observer at some later time t_R. It's enough to consider light that moves radially, so we'll put $\vec{p} \doteq (p^t, p^{\bar{r}}, 0, 0)$.

Goal: Compare energy of light ~~an~~ when it is emitted to the energy when it is received. To do this, we imagine a comoving observer at emission and reception:

$$E_{emit} = -\vec{p}_{emit}\,\vec{u}_{emit}$$
$$= p^t_{emit}\,c$$

Since $u_t = -c$ for a comoving observer.

Now, let's propagate this across spacetime as a radial geodesic and see what energy it gets at $t = t_e$.

Two rules regarding the light:

1: Light-like trajectory, so $\vec{p}\cdot\vec{p} = 0$:

$$-(p^t)^2 + a^2(t)\,(p^{\bar{r}})^2 = 0 \rightarrow p^{\bar{r}} = p^t / a(t)$$

2: Geodesic, so we extremize

$$L = \frac{1}{2}\sum g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

$$= -\frac{c^2}{2}\left(\frac{dt}{d\lambda}\right)^2 + \frac{a^2(t)}{2}\left(\frac{d\bar{r}}{d\lambda}\right)^2$$

Let's focus on the $x^0 = ct$ component:

$$\frac{\partial L}{\partial x^0} = \frac{1}{c}\frac{\partial L}{\partial t} = \frac{1}{c}\, a\dot{a}\left(p^{\bar{r}}\right)^2 , \qquad \dot{a} = \frac{da}{dt}$$

$$\frac{\partial L}{\partial (dx^0/d\lambda)} = -c\,\frac{dt}{d\lambda} = -p^t$$

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial (dx^0/d\lambda)}\right] = -\frac{dp^t}{d\lambda}$$

$$\frac{\partial L}{\partial x^0} - \frac{d}{d\lambda}\frac{\partial L}{\partial (dx^0/d\lambda)} = 0 \longrightarrow$$

$$\frac{a\dot{a}}{c}\left(p^{\bar{r}}\right)^2 + \frac{dp^t}{d\lambda} = 0$$

-or-

$$\frac{1}{c}\frac{\dot{a}}{a}\left(p^t\right)^2 + \frac{dp^t}{d\lambda} = 0$$

But, $\dot{a}\,p^t = \left(\dfrac{da}{dt}\right) c\,\dfrac{dt}{d\lambda} = c\,\dfrac{da}{d\lambda}$

Our equation becomes

$$\frac{da/d\lambda}{a}\,p^t + \frac{dp^t}{d\lambda} = 0$$

-or-

$$\frac{da/d\lambda}{a} = -\frac{dp^t/d\lambda}{p^t}$$

Integrate both sides from $\lambda = \lambda_E$ (corresponding to $t = t_E$) to $\lambda = \lambda_R$ (corresponding to $t = t_R$):

$$\ln\left[\frac{p^t(t_R)}{p^t(t_E)}\right] = -\ln\left[\frac{a(t_R)}{a(t_E)}\right]$$

or $$\frac{p^t(t_R)}{p^t(t_e)} = \frac{a(t_e)}{a(t_R)}$$

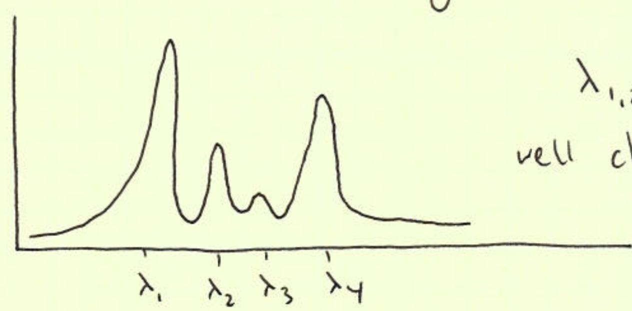
We measure with a comoving observer, and so $E_{rec} = p^t(t_R)\,c$, which gives us

$$\boxed{\frac{E_{Rec}}{E_{emit}} = \frac{a(t_e)}{a(t_R)}}$$

In words, this means that the way energy associated with light behaves gives us a way to directly probe the scale factor of the universe. This is HUGELY important! How do we use it?

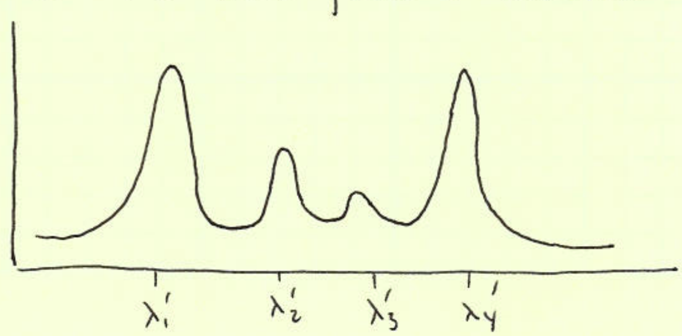
Excited atoms and molecules emit light with distinct spectral lines:

Flux



$\lambda_{1,2,3,4}$: Very well characterized by lab measurements.

Each wavelength corresponds to a distinct energy level: $E = h\nu = hc/\lambda$. When we measure it, the light has been redshifted due to the scale factor's evolution:



$$\frac{\lambda_1'}{\lambda_1} = \frac{\lambda_2'}{\lambda_2} = \frac{\lambda_3'}{\lambda_3} = \frac{\lambda_4'}{\lambda_4} = \frac{a(t_R)}{a(t_E)}$$

$$\equiv 1 + z,$$

where z = "the redshift"

Do this for many sources, build a map of how the scale factor evolves.

Next: How do we get the evolution of the scale factor as a function of time, and how does this relate to the matter content of the universe?