

## Recap of last lecture

- Invariance of speed of light leads to a blending of "space" + "time" as seen by different observers: Moving clocks run slow, moving rulers are shortened.
- Just as observers with different relative orientations differ on which direction is "right" and "forward", observers with different relative velocities differ on "space" and "time". To relate their coords, use a Lorentz transformation:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Origins of the frames match at  $t=0, t'=0$ .

Primed frame moves with  $\underline{v} = v\hat{e}_x = \beta c\hat{e}_x$  as seen by observers at rest in unprimed frame.

Mixing of space and time under Lorentz transformation demonstrates that neither "space" nor "time" is special - only a unification of the two makes sense. We call this unification spacetime.

Nature doesn't care which direction you call "x"!  
 What if the "boost" is along y?

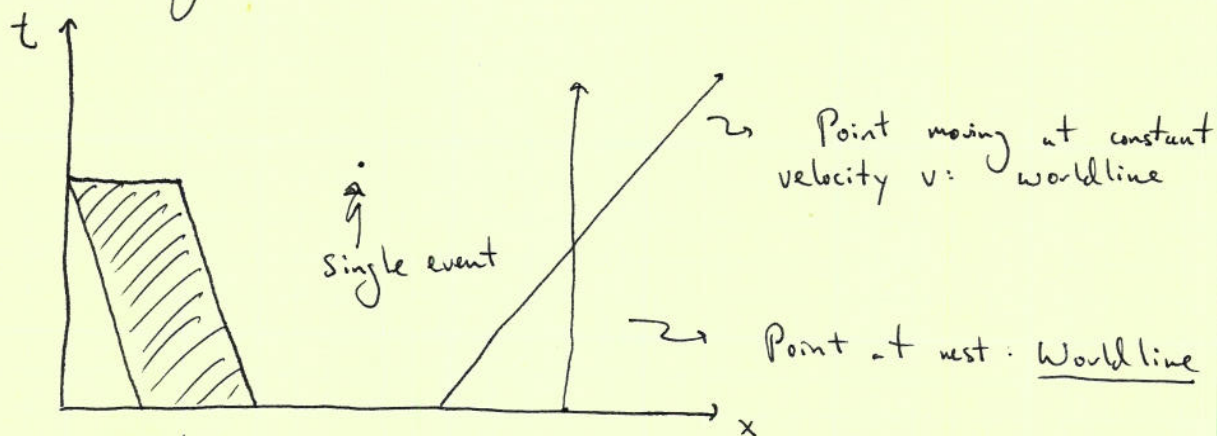
$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Similarly if boost is along z.

If boost is along some arbitrary direction, result can be understood most easily as a rotation plus a boost. Results end up kind of messy, but are still OK.

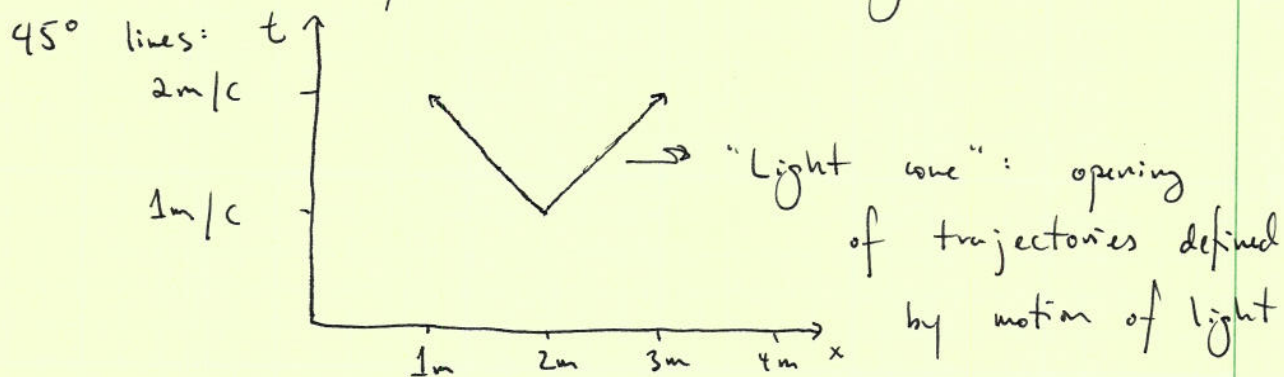
The geometry of spacetime: Focus of today's lecture

Spacetime diagrams: Figures that illustrate how space and time are laid out, as seen by a particular inertial reference frame. Convention: time is vertical, space is horizontal.



↳ Worksheet of an extended object moving at constant negative  $v$ .

Units are traditionally chosen so that light moves on



Useful side note: speed of light is almost exactly equal to 1 foot per nanosecond!

Very common in relativity research to use "light second" as unit of distance. Then,  $c = 1 \text{ lightsecond} / \text{second} = 1$ . Very convenient!



Given axes  $(t, x)$ , how do we draw axes  $(t', x')$  for an observer in a different Lorentz frame?

Look at transformation rule:

$$ct' = \gamma ct - \beta \gamma x$$

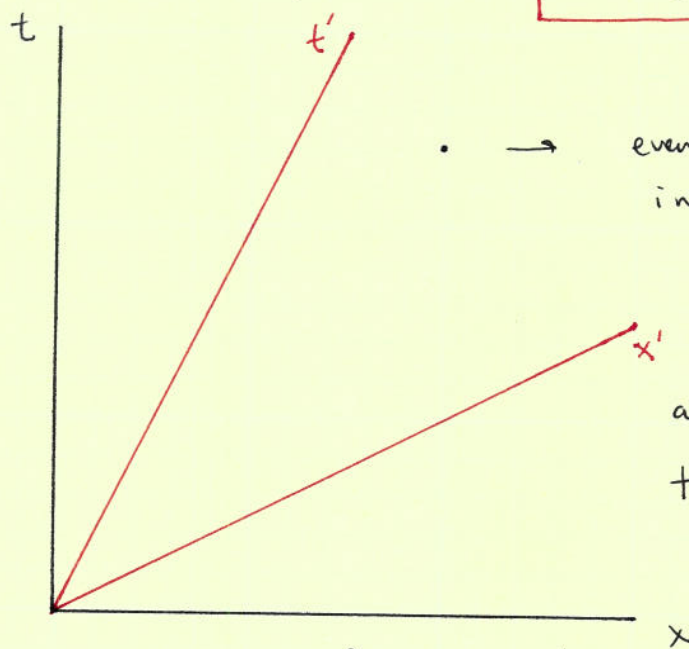
$$x' = -\beta \gamma ct + \gamma x$$

The  $t'$  axis is defined by  $x' = 0$ :

$$0 = -\beta \gamma ct + \gamma x \rightarrow t = \frac{x}{\beta c} = \frac{x}{v}$$

The  $x'$  axis is defined by  $t' = 0$

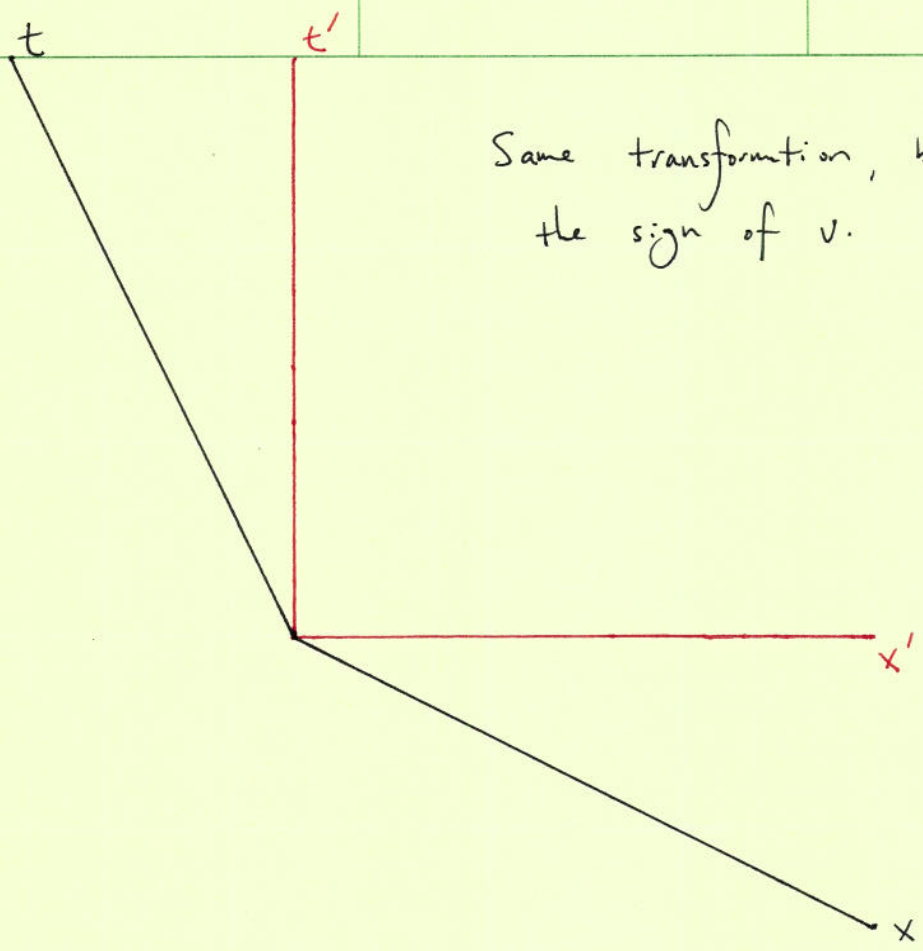
$$0 = \gamma ct - \beta \gamma x \rightarrow t = \frac{\beta x}{c} = \frac{vx}{c^2}$$



- $\rightarrow$  event: same position in spacetime, but different space and time coordinates are assigned in the two reference frames.

$(t', x')$  axes drawn for  $v = c/2$ .

If we use  $(t', x')$  as our rest frame, how do the  $(t, x)$  axes look?



Same transformation, but switch the sign of  $v$ .

Drawing transformed axes in this way illustrates why length contraction and time dilation arise: Events which are simultaneous - occur at the same time - in one frame of reference are NOT simultaneous in another frame. Events which occur in the same location in one frame do not occur in the same location in the other.

This is the essence of how "space" and "time" are mixed, but "spacetime" the joint quantity remains unified. Different observers agree on "spacetime", but split it into "space" and "time" in different ways.

## The invariant interval

Take two events, A + B. Compute their separation in time and space in some given frame:

$$\Delta t = t_B - t_A, \quad \Delta x = x_B - x_A, \quad \Delta y = \dots$$

Compute 
$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

Theorem: All inertial observers, in all reference frames, agree on the value of  $\Delta s^2$ .

Really interesting thing is whether  $\Delta s^2$  is positive, negative, or zero:

$\Delta s^2 < 0$ : Means that the interval is dominated by the  $\Delta t$  piece. Events have TIMELIKE separation.

In this case, there exists some Lorentz frame in which events A + B are at the same location in space, and are separated only by time.

If  $\Delta s^2 < 0$ , then  $\Delta \tau \equiv \sqrt{-\Delta s^2} / c$  is the time elapsed between events A + B in the frame in which the events are co-located.

$\Delta \tau \equiv$  "the proper time interval"

Proper time is the time measured by an observer who is at rest in that frame.



$\Delta s^2 > 0$  : Means interval is dominated by  $\Delta x^2 + \Delta y^2 + \Delta z^2$ .  
 Events have SPACELIKE separation. In this case, there exists a Lorentz frame in which events A and B are simultaneous.

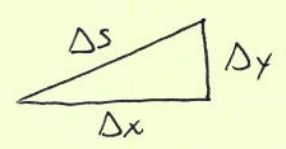
If  $\Delta s^2 > 0$ , then  $\Delta s$  is the distance between A and B in the frame in which they are simultaneous.  
 $\Delta s =$  "Proper separation" or "Proper length"

$\Delta s^2 = 0$  : Means  $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = c \Delta t$

Events A & B can be connected by a light pulse :  
 They have a "lightlike" or "null" separation.

The relation  $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$  generalizes the Pythagorean theorem to spacetime. In space, we have

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$



for triangles on a flat surface. (Not true for triangles on the surface of a sphere, for example.)

In 3-D,  $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$

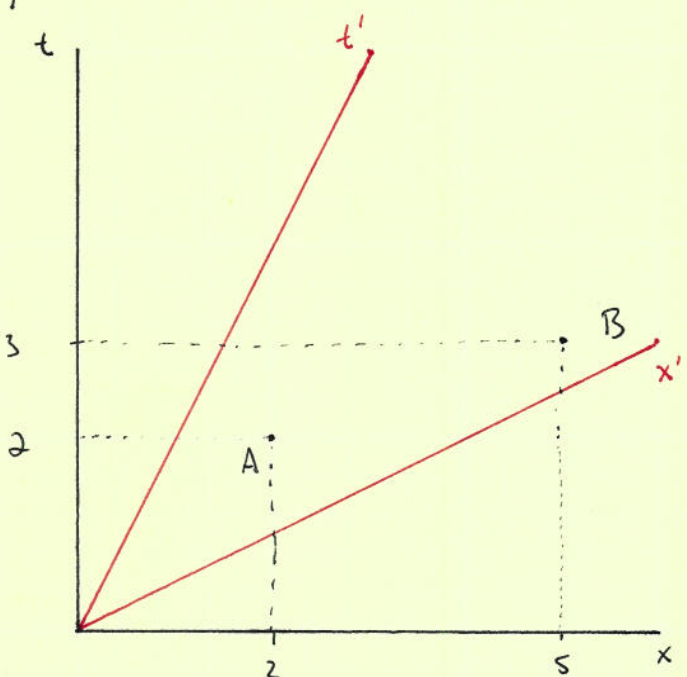
In spacetime,  $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ .

Why the minus sign on the time piece? That's just the geometry of spacetime - it's how the universe is made.

### Geometric objects versus representation of geometric objects.

As emphasized in Lecture 1, geometric objects are entities in spacetime. All observers agree on certain properties of these objects, but different frames may describe them in different ways.

Example: events.



$$v = \frac{c}{2} \hat{e}_x$$

Event A:  $t = 2 \text{ sec}$ ,  $x = 2 \text{ lightseconds}$

Event B:  $t = 3 \text{ sec}$ ,  $x = 5 \text{ lightseconds}$

As seen in other frame

Event A:  $t' = 2/\sqrt{3} \text{ sec}$ ,  $x = 2/\sqrt{3} \text{ lightseconds}$   
 $\approx 1.15 \text{ sec}$   $\approx 1.15 \text{ lightseconds}$

~~Event B:~~

Event B:  $t' = 2\sqrt{3} - 5/\sqrt{3}$ ,  $x' = 10/\sqrt{3} - \sqrt{3}$   
 $\approx 0.577 \text{ sec}$ ,  $\approx 4.04 \text{ lightseconds}$

(Notice the order of the events is different in the two frames!)