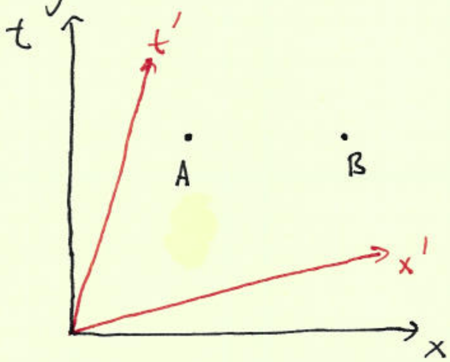
Recap:

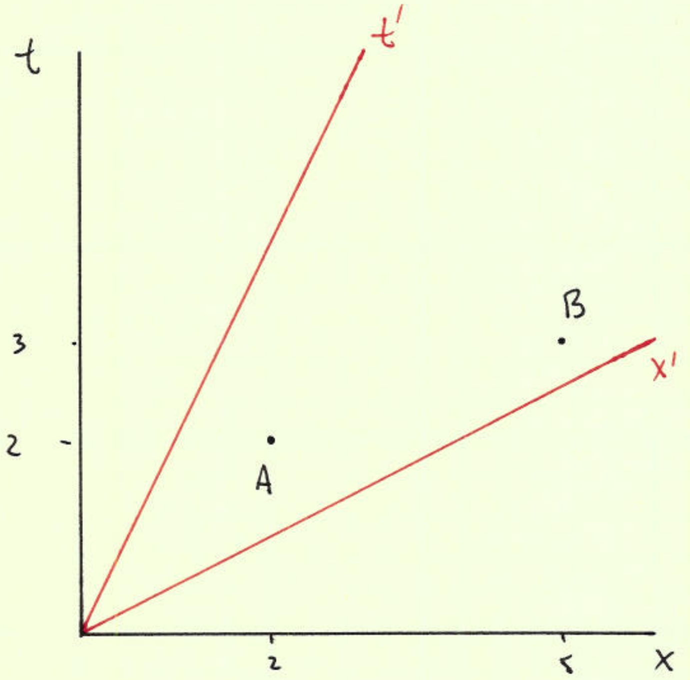
- Geometry of special relativity reveals that Lorentz transformation causes a breakdown of simultaneity: Events that are simultaneous in one inertial reference frame are not simultaneous in another




Events A + B are simultaneous in $(t, x)$ frame, but not in $(t', x')$ frame.

- The "interval" $\Delta s^2 = -c\,\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ is invariant: ALL inertial observers agree on the value of $\Delta s^2$.
    - $\Delta s^2 < 0$ : Events have TIMELIKE separation
    - $\Delta s^2 > 0$ : Events have SPACELIKE separation
    - $\Delta s^2 = 0$ : Events have LIGHTLIKE separation.

Situation we laid out last time:




$$\underset{\sim}{v} = \frac{c}{2}\, \underset{\sim}{e}_x$$

$t_A = 2 \text{ sec}$ $\qquad$ $x_A = 2 \text{ lightsec}$

$t_B = 3 \text{ sec}$ $\qquad$ $x_B = 5 \text{ lightsec}$

$t'_A = \frac{2}{\sqrt{3}} \text{ sec}$ $\qquad$ $x'_A = \frac{2}{\sqrt{3}} \text{ lightsec}$

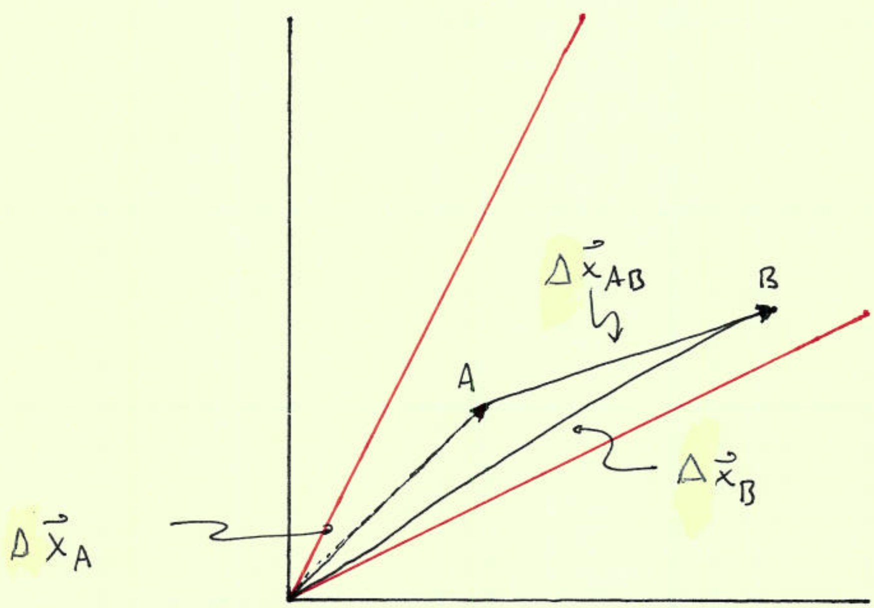
$\simeq 1.15 \text{ sec}$ $\qquad$ $\simeq 1.15 \text{ lightsec}$

$t'_B = \left(2\sqrt{3} - \frac{5}{\sqrt{3}}\right) \text{ sec}$ $\qquad$ $x'_B = \left(\frac{10}{\sqrt{3}} - \sqrt{3}\right) \text{ sec}$

$\simeq 0.577 \text{ sec}$ $\qquad$ $\simeq 4.04 \text{ lightsec}$

Both observers agree on the location of the events in spacetime, but assign the events very different coordinates - don't even agree on which came first! All agree on the value of $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2$ however.

From this, we can construct displacement vectors in spacetime:




These vectors are geometric objects in spacetime: All observers agree on their properties:

$\Delta\vec{x}_A$ points from origin to event A

$\Delta\vec{x}_B$ points from origin to event B

$\Delta\vec{x}_{AB}$ points from event A to event B.

But, they will break them into components in different ways.

In unprimed frame:

$$\Delta \vec{x}_A = \Delta x_A^0 \, \vec{e}_0 + \Delta x_A^1 \, \vec{e}_1 + \Delta x_A^2 \, \vec{e}_2 + \Delta x_A^3 \, \vec{e}_3$$

-or-

$$= \Delta x_A^t \, \vec{e}_t + \Delta x_A^x \, \vec{e}_x + \Delta x_A^y \, \vec{e}_y + \Delta x_A^z \, \vec{e}_z$$

Here, $\vec{e}_0 = \vec{e}_t$ is a unit-vector which points along the t-axis;

$\vec{e}_1 = \vec{e}_x$ is a unit vector which points along the x-axis; etc.

$\Delta x_A^0 = \Delta x_A^t = c \Delta t_A \rightarrow$ displacement along t from origin to event A

$= 2$ light seconds

$\Delta x_A^1 = \Delta x_A^x = \Delta x_A \rightarrow$ displacement along x from origin to event A

$= 2$ light seconds

$$\Delta x^2{}_A = \Delta x^y{}_A = \Delta y_A = 0$$

$$\Delta x^3{}_A = \Delta x^z{}_A = \Delta z_A = 0$$

Likewise, we can write $\Delta \vec{x}_B$ and $\Delta \vec{x}_{AB}$ in terms of components of the unit vectors $\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3$.

In primed frame:

$$\Delta \vec{x}_A = \Delta x_A^{0'} \vec{e}_{0'} + \Delta x_A^{1'} \vec{e}_{1'} + \Delta x_A^{2'} \vec{e}_{2'} + \Delta x_A^{3'} \vec{e}_{3'}$$

-or- $$= \Delta x_A^{t'} \vec{e}_{t'} + \Delta x_A^{x'} \vec{e}_{x'} + \Delta x_A^{y'} \vec{e}_{y'} + \Delta x_A^{z'} \vec{e}_{z'}$$

$\vec{e}_{0'} = \vec{e}_{t'}$ is a unit vector pointing along the $t'$ axis;

$\vec{e}_{1'} = \vec{e}_{x'}$ is a unit vector pointing along the $x'$ axis; etc.

The components in this case are the displacements in the primed coordinates:

$$\Delta x_A^{0'} = \Delta x_A^{t'} = c\Delta t_A' = 2/\sqrt{3} \text{ light seconds}$$

$$\Delta x_A^{1'} = \Delta x_A^{x'} = \Delta x_A' = 2/\sqrt{3} \text{ light seconds}$$

-etc-

Key points: In both cases, the <u>vectors</u> $\Delta \vec{x}_A$ are identical. However, the primed and the unprimed frames break them up into different components, and using different unit vectors.

It's just like how we think about a vector in two coordinate systems related by a rotation. Same vector; different axes, different components.

Writing out

$$\Delta\vec{x}_A = \Delta x^0_A \, \vec{e}_0 + \Delta x^1_A \, \vec{e}_1 + \cdots$$

is cumbersome. So, we introduce indices to give us a shorthand:

$$\Delta\vec{x}_A = \sum_{\mu=0}^{3} \Delta x^\mu_A \, \vec{e}_\mu$$

$$= \sum_{\mu'=0}^{3} \Delta x^{\mu'}_A \, \vec{e}_{\mu'}$$

Convention: If index is a Greek letter, it denotes a spacetime component and runs from 0 to 3. If it is a Latin letter, it denotes a spatial component, and runs from 1 to 3.

Another convention: If two symbols with the same index are next to each other; one has index in "upstairs" position, other in "downstairs" position, then the sum is assumed:

$$\Delta\vec{x}_A = \sum_{\mu=0}^{3} \Delta x^\mu_A \, \vec{e}_\mu \equiv \Delta x^\mu_A \, \vec{e}_\mu$$

"Einstein summation convention"

Key thing: Given that

$$\Delta\vec{x} = \sum_{\mu=0}^{3} \Delta x^{\mu}\, \vec{e}_{\mu}$$

$$= \sum_{\alpha'=0}^{3} \Delta x^{\alpha'}\, \vec{e}_{\alpha'}$$

are the same geometric object, how do we relate the primed and unprimed components, and the primed and unprimed unit vectors?

Components are easy: the components of the displacement vector are simply the difference between different coordinates of events. So, they are related by a Lorentz transformation:

$$\begin{bmatrix} \Delta x^{0'} \\ \Delta x^{1'} \\ \Delta x^{2'} \\ \Delta x^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x^{0} \\ \Delta x^{1} \\ \Delta x^{2} \\ \Delta x^{3} \end{bmatrix}$$

Writing out the matrix is annoying and cumbersome. We use indices to simplify it:

$$\Delta x^{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\alpha'}{}_{\mu} \Delta x^{\mu}$$

$\Lambda^{\alpha'}{}_{\mu}$ ≡ Row $\alpha'$, column $\mu$ of the Lorentz transformation matrix.

Since the inverse of a Lorentz transformation is another Lorentz transformation, it's easy to invert this:

$$\Delta x^{\mu} = \sum_{\alpha'=0}^{3} \Lambda^{\mu}{}_{\alpha'} \Delta x^{\alpha'}$$

$\Lambda^{\mu}{}_{\alpha'}$ ≡ Row $\mu$, column $\alpha'$ of the Lorentz transformation that inverses our previous one.

Convention: Given $\Lambda^{\alpha'}{}_{\mu}$, this is an element of the matrix that transforms from frame of lower index to frame of upper index.

Mathematical expression of the inverse relation in index notation:

$$\sum_{\alpha'=0}^{3} \Lambda^{\mu}{}_{\alpha'} \Lambda^{\alpha'}{}_{\nu} \equiv \delta^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha'} \Lambda^{\alpha'}{}_{\nu}$$

$\delta^{\mu}{}_{\nu}$ is called the Kronecker delta: it is a mathematical object whose value is $1$ if $\mu = \nu$, $0$ if $\mu \neq \nu$

→ In other words, it's the identity matrix!

Equivalent form of this inverse relation:

$$\sum_{\mu=0}^{3} \Lambda^{\alpha'}{}_{\mu} \Lambda^{\mu}{}_{\beta'} \equiv \delta^{\alpha'}{}_{\beta'} = \Lambda^{\alpha'}{}_{\mu} \Lambda^{\mu}{}_{\beta'}$$

With all this in mind, let's infer what the transformation rule is for the unit vectors.

- We already know the components transform as $\Delta x_A^{\alpha'} = \Lambda^{\alpha'}{}_{\mu} \Delta x_A^{\mu}$

- Let us assume that <u>some</u> matrix $M$ relates the unit vectors $\vec{e}_{\mu}$ and $\vec{e}_{\alpha'}$:

$$\vec{e}_{\alpha'} = M^{\mu}{}_{\alpha'} \vec{e}_{\mu}$$

We also know that the 4-vector $\Delta\vec{x}_A$ is the same geometric object no matter how we represent it:

$$\Delta\vec{x}_A = \sum_{\mu=0}^{3} \Delta x_A^{\mu}\, \vec{e}_{\mu}$$

$$= \sum_{\alpha'=0}^{3} \Delta x_A^{\alpha'}\, \vec{e}_{\alpha'}$$

Plug in the transformation rules to this final line:

$$\Delta\vec{x}_A = \sum_{\alpha'=0}^{3} \left(\sum_{\mu=0}^{3} \Lambda^{\alpha'}{}_{\mu}\, \Delta x_A^{\mu}\right)\left(\sum_{\nu=0}^{3} M^{\nu}{}_{\alpha'}\, \vec{e}_{\nu}\right)$$

- Notice: very careful to make sure that the indices we sum over are all different!

Next, we exchange the order of the sums:

$$\Delta\vec{x}_A = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \left(\sum_{\alpha'=0}^{3} \Lambda^{\alpha'}{}_{\mu}\, M^{\nu}{}_{\alpha'}\right) \Delta x_A^{\mu}\, \vec{e}_{\nu}$$

Examining this expression, we see that it will work out perfectly <u>IF</u>

$$\sum_{\alpha'=0}^{3} \Lambda^{\alpha'}{}_{\mu}\, M^{\nu}{}_{\alpha'} = \delta^{\nu}{}_{\mu}$$

In this case, we would have

$$\Delta\vec{x}_A = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \delta^{\nu}{}_{\mu}\, \Delta x^{\mu}{}_{A}\, \vec{e}_{\nu}$$

$$= \sum_{\mu=0}^{3} \Delta x^{\mu}{}_{A}\, \vec{e}_{\mu}$$

→ Exactly what it should be!

Key to making all of this work is the relationship

$$\sum_{\alpha'=0}^{3} \Lambda^{\alpha'}{}_{\mu} M^{\nu}{}_{\alpha'} = \delta^{\nu}{}_{\mu}$$

This relationship implies that $M^{\nu}{}_{\alpha'} = \Lambda^{\nu}{}_{\alpha'}$ – the matrix that inverts the Lorentz transformation $\Lambda^{\alpha'}{}_{\mu}$.

With this we now have a complete "glossary" for how to transform both vector components and unit vectors:

$$\Delta x_A^{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\alpha'}{}_{\mu} \Delta x_A^{\mu}$$

$$\vec{e}_{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\mu}{}_{\alpha'} \vec{e}_{\mu}$$

–or–

$$\Delta x_A^{\alpha'} = \Lambda^{\alpha'}{}_{\mu} \Delta x_A^{\mu}$$

$$\vec{e}_{\alpha'} = \Lambda^{\mu}{}_{\alpha'} \vec{e}_{\mu}$$

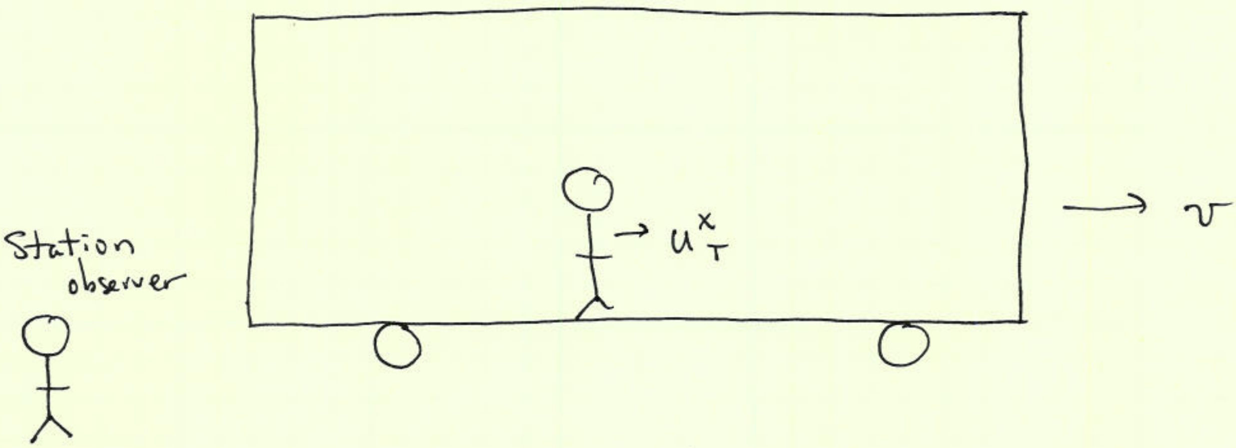
At the end of the day, the rule is simple: just "Line up the indices."

Kinematics in special relativity

We've done a lot to understand length, time, and geometry - but that's just a start to setting up the physics we want to describe.

First step: How do velocities transform?

Consider someone on board a train, walking with speed $u_T^x$ as measured by stationary observers on the train. In other words, in a time interval $\Delta t_T$, this person moves a distance $\Delta x_T = u_T^x \Delta t_T$.




How do observers in the station measure this? Use Lorentz transformation to find out!

$$u_S^x = \frac{\Delta x_S}{\Delta t_S} = \frac{\gamma(\Delta x_T + v\Delta t_T)}{\gamma(\Delta t_T + v\Delta x_T/c^2)}$$

$$= \frac{(\Delta x_T/\Delta t_T + v)}{1 + \frac{v}{c^2}\frac{\Delta x_T}{\Delta t_T}}$$

$$\rightarrow \boxed{u_S^x = \frac{u_T^x + v}{1 + u_T^x v/c^2}}$$

With this formula, can never add sub-light speeds to get a speed that exceeds light.

Example: $u_T^x = 0.9c$ $\quad v = 0.9c$

$$u_S^x = \frac{0.9c + 0.9c}{1 + 0.81} = 0.9945\,c$$

Enforces $c$ as nature's "speed limit."

How do velocity components perpendicular to the frames' relative motion transform? Again, use Lorentz:

$$u_S^y = \frac{\Delta y_S}{\Delta t_S} = \frac{\Delta y_T}{\gamma\left(\Delta t_T + v\,\Delta x_T / c^2\right)}$$

Note: Assuming motion may have components along frame's relative motion.

$$\rightarrow \boxed{u_S^y = \frac{u_T^y}{\gamma\left(1 + u_T^x v / c^2\right)}, \quad u_S^z = \frac{u_T^z}{\gamma\left(1 + u_T^x v / c^2\right)}}$$

Note: $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$, $v$ = relative speed of two frames.

$\rightarrow$ $\gamma$ does not involve $\underset{\sim}{u}$!